High-Order Random Walks and Generalized Laplacians on Hypergraphs

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Abstract. Despite the extreme success of spectral graph theory, there are relatively few papers applying spectral analysis to hypergraphs. Chung first introduced Laplacians for regular hypergraphs and showed some useful applications. Other researchers have treated hypergraphs as weighted graphs and then studied the Laplacians of the corresponding weighted graphs. In this paper, we aim to unify these very different versions of Laplacians for hypergraphs. We introduce a set of Laplacians for hypergraphs through studying high-order random walks on hypergraphs. We prove that the eigenvalues of these Laplacians can effectively control the mixing rate of high-order random walks, the generalized distances/diameters, and the edge expansions.

1. Introduction

Many complex networks (such as spatial networks [Demir et al. 08], cellular networks [Klamt et al. 09], and biomolecular networks [Zhang 07]) have richer structures than graphs can have. Inherently, they have hypergraph structures: interconnections often cross multiple nodes. Treating these networks as graphs causes a loss of structure. Nonetheless, it is still popular to use graph tools to

study these networks; one of them is the Laplacian spectrum. Let G be a graph on n vertices. The *Laplacian* $\mathcal{L}$ of G is the $n \times n$ matrix $I - T^{-1/2} A T^{-1/2}$, where A is the adjacency matrix and T is the diagonal matrix of degrees. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}$, indexed in nondecreasing order. It is known that $0 \leq \lambda_i \leq 2$ for $0 \leq i \leq n-1$. If G is connected, then $\lambda_1 > 0$. The first nonzero Laplacian eigenvalue λ_1 is related to many graph parameters, such as the mixing rate of random walks, the graph diameter, the neighborhood expansion, the Cheeger constant, the isoperimetric inequalities, expander graphs, and quasirandom graphs [Aldous and Fill 12, Alon 86, Chung 89, Chung et al. 94, Chung 97, Lawler and Sokal 88, Mihail 89].

In this paper, we define a set of Laplacians for hypergraphs. Laplacians for regular hypergraphs were first introduced in [Chung 93], which used the homology approach. The first nonzero Laplacian eigenvalue can be used to derive several useful isoperimetric inequalities. It seems hard to extend Chung's definition to general hypergraphs. There are some researchers who have treated a hypergraph as a multiedge graph and then defined its Laplacian to be the Laplacian of a corresponding multiedge graph. For example, it was shown in [Rodríguez 09] that the approach above has some applications to bisections, the average minimal cut, the isoperimetric number, the max-cut, the independence number, the diameter, etc. Rodríguez's approach was introduced in [Li 04]; Ramanujan hypergraphs were defined in [Li and Solé 96]. Another direction for studying the spectra of adjacency matrices of hypergraphs is to use the notion of tensor product. This direction was taken in [Friedman and Wigderson 95] to investigate spectra of adjacency matrices of 3-uniform hypergraphs. Recently, spectra of adjacency hypermatrices of r-uniform hypergraphs were considered in [Cooper and Dutle 12].

What are the "right" Laplacians for hypergraphs? To answer this question, let us recall how the Laplacian was introduced in graph theory. One approach uses a geometric/homological analogue, where the Laplacian is defined as a self-adjoint operator on the functions over vertices. Another approach uses random walks, where the Laplacian is the symmetrization of the transition matrix of the random walk on a graph. In [Chung 93], the first approach was taken, and Chung defined her Laplacians for regular hypergraphs. In this paper, we take the second approach and define the Laplacians through high-order random walks on hypergraphs.

A high-order walk on a hypergraph H can be roughly viewed as a sequence of overlapped oriented edges $F_1, F_2, \ldots, F_k$. For $1 \leq s \leq r-1$, we say that $F_1, F_2, \ldots, F_k$ is an s-walk if $|F_i \cap F_{i+1}| = s$ for each i in $\{1, 2, 3, \ldots, k-1\}$. For example, a *Hamiltonian s-cycle* is a special s-walk that covers each vertex exactly once. There are many papers [Dudek and Frieze 11, Dudek and Frieze 12, Frieze 10, Hàn and Schacht 99, Katona and Kierstead 99, Keevash

et al. 12, Kühn et al. 12, Kühn and Osthus 06, Rödl et al. 06, Rödl et al. 08] studying Hamiltonian s-cycles in hypergraphs. A detailed definition of high-order random walks will be given later.

From this rough view of high-order random walks, if we refer to the intersection of two consecutive edges as a stop, then it is possible that there is some vertex from the stop contained in more than two consecutive edges when $s > r/2$, so we need to specify the edge to which the vertex from the stop belongs. Hence we need to assume a stop to be an ordered set of s vertices, which causes an s-walk to be reversible when $1 \leq s \leq r/2$ and irreversible when $s > r/2$. Therefore, in the definition of the sth Laplacian for r-uniform hypergraphs, we will split the definition into two cases. The loose case is that in which $1 \leq s \leq r/2$; in this case we will reduce the sth Laplacian of a hypergraph to the Laplacian of an associated weighted undirected graph. The tight case is that in which $r/2 < s \leq r - 1$; in this case we will reduce the sth Laplacian of a hypergraph to the Laplacian of an associated Eulerian directed graph.

Before we give the definition of the sth Laplacian of a hypergraph, we need to review necessary results on the Laplacian of a weighted undirected graph and the Laplacian of an Eulerian directed graph. The choice of s enables us to define a set of Laplacian matrices $\mathcal{L}^{(s)}$ for H. For $s = 1$, our definition of Laplacian $\mathcal{L}^{(1)}$ is the same as the definition in [Rodríguez 09]. For $s = r - 1$, while we restrict our attention to regular hypergraphs, our definition of the Laplacian $\mathcal{L}^{(r-1)}$ is similar to the definition in [Chung 93]. We will discuss the relationship between these notions in the final section of this paper. Our definition of Laplacians is also closely related to the singular values used in the unpublished work [Butler unpubl.].

In this paper, we show several applications of the Laplacians of hypergraphs, such as the mixing rate of high-order random walks, generalized diameters, and edge expansions. Our approach allows users to select a "right" Laplacian to fit their special requirements.

The rest of the paper is organized as follows. In Section 2, we review and prove some useful results on the Laplacians of weighted graphs and Eulerian directed graphs. The definition of Laplacians for hypergraphs will be given in Section 3. We will prove some properties of the Laplacians of hypergraphs in Section 4, and consider several applications in Section 5. In the final section, we will comment on future directions.

2. Preliminary Results

In this section, we review some results on Laplacians of weighted graphs and Eulerian directed graphs. The results on Laplacians of weighted graphs will be applied

to the sth Laplacians of hypergraphs for $1 \leq s \leq r/2$, and those on Laplacians of Eulerian directed graphs will be applied to the sth Laplacians of hypergraphs for $r/2 < s \leq r - 1$.

In this paper, we frequently switch domains from hypergraphs to weighted (undirected) graphs and/or to directed graphs. To reduce confusion, we use the following conventions throughout this paper. We denote a weighted graph by G, a directed graph by D, and a hypergraph by H. The sets of vertices are denoted by $V(G)$, $V(D)$, and $V(H)$, respectively. (Whenever it is clear from the context, we will write such a set as V for short.) The edge sets are denoted by $E(G)$, $E(D)$, and $E(H)$, respectively. The degrees d_* and volumes $\mathrm{vol}(*)$ are defined separately for the weighted graph G, for the directed graph D, and for the hypergraph H. Readers are warned to interpret them carefully depending on the context.

For a positive integer s and a vertex set V, let $V^{\underline{s}}$ be the set of all (ordered) s-tuples consisting of s distinct elements in V. Let $\binom{V}{s}$ be the set of all unordered (distinct) s-subsets of V.

Let $\mathbf{1}$ be the row (or the column) vector with all entries of value 1, and let I be the identity matrix. For a row (or column) vector f, the norm $\|f\|$ is always the L_2-norm of f.

2.1.   Laplacians of Weighted Graphs

A *weighted graph* G on the vertex set V is an undirected graph associated with a weight function $w : V \times V \to \mathbb{R}^{\geq 0}$ satisfying $w(u,v) = w(v,u)$ for all u and v in $V(G)$. Here we always assume $w(v,v) = 0$ for every $v \in V$.

A simple graph can be viewed as a special weighted graph such that each edge has weight 1 and each non-edge has weight 0. Many concepts of simple graphs are naturally generalized to weighted graphs. If $w(u,v) > 0$, then u and v are *adjacent*, written as $x \sim y$. The graph distance $d(u,v)$ between two vertices u and v in G is the minimum integer k such that there is a path $u = v_0, v_1, \ldots, v_k = v$ in which $w(v_{i-1}, v_i) > 0$ for $1 \leq i \leq k$. If no such k exists, then we let $d(u,v) = \infty$. If the distance $d(u,v)$ is finite for every pair (u,v), then G is *connected*. For a connected weighted graph G, the *diameter* (denoted by $\mathrm{diam}(G)$) is the largest value of $d(u,v)$ among all pairs of vertices (u,v).

The *adjacency matrix* A of G is defined as the matrix of weights, i.e., $A(x,y) = w(x,y)$ for all x and y in V. The *degree* d_x of a vertex x is $\sum_y w(x,y)$. Let T be the diagonal matrix of degrees in G. The *Laplacian* $\mathcal{L}$ is the matrix $I - T^{-1/2} A T^{-1/2}$. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of $\mathcal{L}$, indexed in nondecreasing order. It is known [Chung 97] that $0 \leq \lambda_i \leq 2$ for $0 \leq i \leq n - 1$. If G is connected, then $\lambda_1 > 0$.

From now on, we assume that G is connected. The first nontrivial Laplacian eigenvalue λ_1 is the most useful one. It can be written in terms of the Rayleigh quotient as follows (see [Chung 97]):

$$\lambda_1 = \inf_{f \perp T\mathbf{1}} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x,y)}{\sum_x f(x)^2 d_x}. \tag{2.1}$$

Here the infimum is taken over all functions $f : V \to \mathbb{R}$, which is orthogonal to the degree vector $\mathbf{1}T = (d_1, d_2, \ldots, d_n)$. Similarly, the largest Laplacian eigenvalue λ_{n-1} can be defined in terms of the Rayleigh quotient as follows:

$$\lambda_{n-1} = \sup_{f \perp T\mathbf{1}} \frac{\sum_{x \sim y} (f(x) - f(y))^2 w(x,y)}{\sum_x f(x)^2 d_x}. \tag{2.2}$$

Note that scaling the weights by a constant factor will not affect the Laplacian. A weighted graph G is *complete* if $w(u,v) = c$ for some constant c such that $c > 0$, independent of the choice of (u,v) with $u \neq v$. We say that G is *bipartite* if there is a partition $V = L \cup R$ such that $w(x,y) = 0$ for all $x, y \in L$ and all $x, y \in R$.

We have the following facts (see [Chung 97]):

1. $0 \leq \lambda_i \leq 2$ for each $0 \leq i \leq n - 1$.

2. The number of 0 eigenvalues equals the number of connected components in G. If G is connected, then $\lambda_1 > 0$.

3. $\lambda_{n-1} = 2$ if and only if G has a connected component that is a bipartite weighted subgraph.

4. $\lambda_{n-1} = \lambda_1$ if and only if G is a complete weighted graph.

It turns out that λ_1 and λ_{n-1} are related to many graph parameters, such as the mixing rate of random walks, the diameter, the edge expansions, and the isoperimetric inequalities.

A random walk on a weighted graph G is a sequence of vertices $v_0, v_1, \ldots, v_k$ such that the conditional probability $\mathbf{Pr}(v_{i+1} = v \mid v_i = u)$ is equal to $w(u,v)/d_u$ for $0 \leq i \leq k - 1$. A *vertex probability distribution* is a map $f : V \to \mathbb{R}$ such that $f(v) \geq 0$ for each v in G and $\sum_{v \in V} f(v) = 1$. It is convenient to write a vertex probability distribution as a row vector. A random walk maps a vertex probability distribution to a vertex probability distribution through multiplying by a transition matrix P from the right, where $P(u,v) = w(u,v)/d_u$ for each pair of vertices u and v. We can write $P = T^{-1}A = T^{-1/2}(I - \mathcal{L})T^{1/2}$. The spectral gap $\bar{\lambda}(P)$, denoted by $\bar{\lambda}$ for short, is $\max\{|1 - \lambda_1|, |1 - \lambda_{n-1}|\}$. Let $\pi(u) = d_u/\mathrm{vol}(G)$ for each vertex u in G. Observe that π is the stationary distribution of the random walk, i.e., $\pi P = \pi$. A random walk is *mixing* if $\lim_{i \to \infty} f_0 P^i = \pi$ for every initial vertex probability distribution f_0. It is known that a random walk is always

mixing if G is connected and not a bipartite graph. To overcome the difficulty resulting from being a bipartite graph (where $\lambda_{n-1} = 2$) for $0 \leq \alpha \leq 1$, we consider an α-lazy random walk, whose transition matrix P_α is given by $P_\alpha(u, u) = \alpha$ for each u and $P_\alpha(u, v) = (1 - \alpha)w(u, v)/d_u$ for each pair of vertices u and v with $u \neq v$. Note that the transition matrix is

$$P_\alpha = \alpha I + (1 - \alpha)T^{-1}A = T^{-1/2}(I - (1 - \alpha)\mathcal{L})T^{1/2}.$$

Let

$$L_\alpha = T^{1/2}P_\alpha T^{-1/2} = I - (1 - \alpha)\mathcal{L}$$

and

$$\bar{\lambda}_\alpha = \max\{|1 - (1 - \alpha)\lambda_1|, |1 - (1 - \alpha)\lambda_{n-1}|\}.$$

Since L_α is a symmetric matrix, we have

$$\bar{\lambda}_\alpha = \max_{u \perp T^{1/2}\mathbf{1}} \frac{\|L_\alpha u\|}{\|u\|}.$$

It turns out that the mixing rate of an α-lazy random walk is determined by $\bar{\lambda}_\alpha$.

Theorem 2.1. *For $0 \leq \alpha \leq 1$, the vertex probability distribution f_k of the α-lazy random walk at time k converges to the stationary distribution π in probability. In particular, we have*

$$\left\| (f_k - \pi)T^{-1/2} \right\| \leq \bar{\lambda}^k \left\| (f_0 - \pi)T^{-1/2} \right\|.$$

Here f_0 is the initial vertex probability distribution.

Proof. Notice that $f_k = f_0 P_\alpha^k$ and $(f_0 - \pi)T^{-1/2} \perp \mathbf{1}T^{1/2}$. We have

$$\left\| (f_k - \pi)T^{-1/2} \right\| = \left\| \left(f_0 P_\alpha^k - \pi P_\alpha^k\right) T^{-1/2} \right\| = \left\| (f_0 - \pi)P_\alpha^k T^{-1/2} \right\|$$
$$= \left\| (f_0 - \pi)T^{-1/2} L_\alpha^k \right\| \leq \bar{\lambda}_\alpha^k \left\| (f_0 - \pi)T^{-1/2} \right\|.$$

This completes the proof. □

For each subset X of $V(G)$, the *volume* $\mathrm{vol}(X)$ is $\sum_{x \in X} d_x$. If $X = V(G)$, then we write $\mathrm{vol}(G)$ instead of $\mathrm{vol}(V(G))$. We have

$$\mathrm{vol}(G) = \sum_{i=1}^{n} d_i = 2 \sum_{u \sim v} w(u, v).$$

If $\bar{X}$ is the complement of X, then we have $\mathrm{vol}(\bar{X}) = \mathrm{vol}(G) - \mathrm{vol}(X)$. For every pair of subsets X and Y of $V(G)$, the *distance* $d(X, Y)$ between X and Y is $\min\{d(x, y) : x \in X, y \in Y\}$.

Theorem 2.2. [Chung 89, Chung 97] *In a weighted graph G, for $X, Y \subseteq V(G)$ with distance at least 2, we have*

$$d(X, Y) \leq \left\lceil \log \sqrt{\frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}} \middle/ \log \frac{\lambda_{n-1} + \lambda_1}{\lambda_{n-1} - \lambda_1} \right\rceil.$$

A special case of Theorem 2.2 is that both X and Y are single vertices, which gives an upper bound on the diameter of G.

Corollary 2.3. [Chung 97] *If G is not a complete weighted graph, then we have*

$$\mathrm{diam}(G) \leq \left\lceil \log \frac{\mathrm{vol}(G)}{\delta} \middle/ \log \frac{\lambda_{n-1} + \lambda_1}{\lambda_{n-1} - \lambda_1} \right\rceil,$$

where δ is the minimum degree of G.

For $X, Y \subseteq V(G)$, let $E(X, Y)$ be the set of edges between X and Y. Namely, we have

$$E(X, Y) = \{(u, v) : u \in X, v \in Y \text{ and } uv \in E(G)\}.$$

We have the following theorem.

Theorem 2.4. [Chung 89, Chung 97] *If X and Y are two subsets of $V(G)$, then we have*

$$\left| |E(X, Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leq \bar{\lambda} \cdot \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(G)}.$$

2.2. Laplacians of Eulerian Directed Graphs

The Laplacian of a general directed graph was introduced in [Chung 05, Chung 06]. The theory is considerably more complicated than the one for undirected graphs, but when we consider a special class of directed graphs, namely Eulerian directed graphs, it turns out to be quite neat.

Let D be a directed graph with vertex set $V(D)$ and edge set $E(D)$. A directed edge from x to y is denoted by an ordered pair (x, y) or by $x \to y$. The *out-neighborhood* $\Gamma^+(x)$ of a vertex x in D is the set $\{y : (x, y) \in E(D)\}$. The

out-degree d_x^+ is $|\Gamma^+(x)|$. Similarly, the *in-neighborhood* $\Gamma^-(x)$ is $\{y : (y,x) \in E(D)\}$, and the *in-degree* d_x^- is $|\Gamma^-(x)|$. A directed graph D is *Eulerian* if $d_x^+ = d_x^-$ for every vertex x. In this case, we simply write $d_x = d_x^+ = d_x^-$ for each x. For a vertex subset S, the *volume* of S, denoted by $\mathrm{vol}(S)$, is $\sum_{x \in S} d_x$. In particular, we write $\mathrm{vol}(D) = \sum_{x \in V} d_x$.

Eulerian directed graphs have many nice properties. For example, an Eulerian directed graph is strongly connected if and only if it is weakly connected.

The *adjacency matrix* of D is a square matrix A satisfying $A(x,y) = 1$ if $(x,y) \in E(D)$ and $A(x,y) = 0$ otherwise. Let T be the diagonal matrix with $T(x,x) = d_x$ for each $x \in V(D)$. Let $\vec{\mathcal{L}} = I - T^{-1/2}AT^{-1/2}$, i.e.,

$$\vec{\mathcal{L}}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -1/\sqrt{d_x d_y} & \text{if } x \to y, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Note that $\vec{\mathcal{L}}$ is not symmetric. We define the Laplacian $\mathcal{L}$ of D to be the symmetrization of $\vec{\mathcal{L}}$, that is,

$$\mathcal{L} = \frac{\vec{\mathcal{L}} + \vec{\mathcal{L}}'}{2}.$$

Since $\mathcal{L}$ is symmetric, its eigenvalues are real and can be listed as $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ in nondecreasing order. Note that λ_1 can also be written in terms of the Rayleigh quotient (see [Chung 05]) as follows:

$$\lambda_1 = \inf_{f \perp T\mathbf{1}} \frac{\sum_{x \to y} (f(x) - f(y))^2}{2 \sum_x f(x)^2 d_x}. \tag{2.4}$$

In [Chung 06], there is a general theorem on the relationship between λ_1 and the diameter. After restricting to Eulerian directed graphs, it can be stated as follows.

Theorem 2.5. [Chung 06] *Suppose D is a connected Eulerian directed graph. Then the diameter of D satisfies*

$$\mathrm{diam}(D) \leq \left\lfloor \frac{2\log(\mathrm{vol}(G)/\delta)}{\log \frac{2}{2-\lambda_1}} \right\rfloor + 1,$$

where λ_1 is the first nontrivial eigenvalue of the Laplacian matrix, and δ is the minimum degree $\min\{d_x \mid x \in V(D)\}$.

The main idea in the proof of this theorem is to use α-lazy random walks on D. A random walk on an Eulerian directed graph D is a sequence of vertices

$v_0, v_1, \ldots, v_k$ such that for $0 \le i \le k - 1$, the conditional probability $\mathbf{Pr}(v_{i+1} = v \mid v_i = u)$ equals $1/d_u$ for each $v \in \Gamma^+(u)$ and is 0 otherwise. For $0 \le \alpha \le 1$, α-lazy random walks are defined similarly. The transition matrix P_α of an α-lazy random walk satisfies

$$P_\alpha = \alpha I + (1 - \alpha) T^{-1} A = T^{-1/2} (I - (1 - \alpha) \vec{\mathcal{L}}) T^{1/2}.$$

In [Chung 05], only 1/2-lazy random walks are considered. Here we prove some results on α-lazy random walks for $\alpha \in [0, 1)$.

Let $\pi(u) = d_u / \mathrm{vol}(D)$ for each $u \in V(D)$. Note that π is the stationary distribution, i.e., $\pi P_\alpha = \pi$. Let

$$L_\alpha = \alpha I + (1 - \alpha) T^{-1/2} A T^{-1/2} = I - (1 - \alpha) \vec{\mathcal{L}} = T^{1/2} P_\alpha T^{-1/2}.$$

The key observation is that there is a unit vector ϕ_0 such that ϕ_0 is a row eigenvector L_α and ϕ_0^T is a column eigenvector of L_α for the largest eigenvalue 1. Here let

$$\phi_0 = \mathbf{1} \cdot \frac{T^{1/2}}{\mathrm{vol}(D)} = \frac{1}{\mathrm{vol}(G)} \left(\sqrt{d_1}, \ldots, \sqrt{d_n} \right).$$

We have

$$\phi_0 L_\alpha = \phi_0 \quad \text{and} \quad L_\alpha \phi_0' = \phi_0'.$$

Let $\phi_0^\perp$ be the orthogonal complement of ϕ_0 in R^n. It is easy to check that L_α maps $\phi_0^\perp$ to $\phi_0^\perp$. Let σ_α be the spectral norm of L_α when restricted to $\phi_0^\perp$. An equivalent definition of σ_α is the second-largest singular value of L_α, i.e.,

$$\sigma_\alpha = \max_{f \perp \phi_0'} \frac{\|L_\alpha f\|}{\|f\|}.$$

Lemma 2.6. *We have the following properties for σ_α:*

1. *For every $\beta \in \phi_0^\perp$, we have $\|L_\alpha \beta\| \le \sigma_\alpha \|\beta\|$.*

2. $(1 - \lambda_1)^2 \le \sigma_0^2 \le 1$.

3. $\sigma_\alpha^2 \le \alpha^2 + 2\alpha(1 - \alpha)\lambda_1 + (1 - \alpha)^2 \sigma_0^2$.

Proof. Item 1 is from the definition of σ_α. Since the largest eigenvalue of L_α is 1, we have $\sigma_\alpha \le 1$. In particular, $\sigma_0^2 \le 1$. Note that $L_0 = T^{-1/2} A T^{-1/2}$. Let $f = g T^{1/2}$. It follows that

$$\sigma_0^2 = \sup_{f \perp \phi_0'} \frac{\|L_0 f\|^2}{\|f\|^2} = \sup_{g \perp T\mathbf{1}} \frac{g' A' T^{-1} A g}{g' T g}.$$

Choose $g \in (T\mathbf{1})^{\perp}$ such that the Rayleigh quotient (2.4) reaches its minimum at g, i.e.,

$$\lambda_1 = \frac{\sum_{x \to y} (g(x) - g(y))^2}{2 \sum_x g(x)^2 d_x}.$$

We have

$$\begin{aligned}
\frac{g'A'T^{-1}Ag}{g'Tg} &= \frac{\sum_x \frac{1}{d_x} \left(\sum_{y \in \Gamma^+(x)} g(y) \right)^2}{\sum_x d_x g(x)^2} \\
&= \frac{\sum_x d_x g(x)^2 \sum_x \frac{1}{d_x} \left(\sum_{y \in \Gamma^+(x)} g(y) \right)^2}{(\sum_x d_x g(x)^2)^2} \\
&\geq \frac{\left(\sum_x g(x) \sum_{y \in \Gamma^+(x)} g(y) \right)^2}{(\sum_x d_x g(x)^2)^2} \\
&= \left(\frac{\sum_x g(x) \sum_{y \in \Gamma^+(x)} g(y)}{\sum_x d_x g(x)^2} \right)^2 = (1 - \lambda_1)^2.
\end{aligned}$$

In the last step, we have used the following argument:

$$\begin{aligned}
\frac{\sum_x g(x) \sum_{y \in \Gamma^+(x)} g(y)}{\sum_x d_x g(x)^2} &= \frac{\frac{1}{2} \sum_{x \to y} \left(g(x)^2 + g(y)^2 - (g(x) - g(y))^2 \right)}{\sum_x d_x g(x)^2} \\
&= 1 - \frac{\sum_{x \to y} (g(x) - g(y))^2}{2 \sum_x d_x g(x)^2} = 1 - \lambda_1.
\end{aligned}$$

Since σ_0 is the maximum over all $g \perp T\mathbf{1}$, we get $(1 - \lambda_1)^2 \leq \sigma_0^2$.

For item 3, we have

$$\begin{aligned}
\sigma_\alpha^2 = \sup_{f \perp \phi_0'} \frac{\|L_\alpha f\|^2}{\|f\|^2} &= \sup_{g \perp T\mathbf{1}} \frac{g'P_\alpha' T P_\alpha g}{g'Tg} \\
&\leq \alpha^2 + \alpha(1 - \alpha) \sup_{g \perp T\mathbf{1}} \frac{g'(A + A')g}{g'Tg} + (1 - \alpha)^2 \sup_{g \perp T\mathbf{1}} \frac{g'A'T^{-1}Ag}{g'Tg} \\
&= \alpha^2 + 2\alpha(1 - \alpha)(1 - \lambda_1) + (1 - \alpha)^2 \sigma_0^2.
\end{aligned}$$

This completes the proof.   □

Theorem 2.7. *For $0 < \alpha < 1$, the vertex probability distribution f_k of an α-lazy random walk on an Eulerian directed graph D at time k converges to the stationary distribution π in probability. In particular, we have*

$$\left\| (f_k - \pi)T^{-1/2} \right\| \leq \sigma_\alpha^k \left\| (f_0 - \pi)T^{-1/2} \right\|.$$

Here f_0 is the initial vertex probability distribution.

The proof is omitted, since it is very similar to the proof of Theorem 2.1. Notice that when $0 < \alpha < 1$, we have $\sigma_\alpha < 1$ by Lemma 2.6. The α-lazy random walk converges to the stationary distribution exponentially fast.

For two vertex subsets X and Y of $V(D)$, let $E(X, Y)$ be the number of directed edges from X to Y, i.e., $E(X, Y) = \{(u, v) : u \in X \text{ and } v \in Y\}$. We have the following theorem on edge expansions in Eulerian directed graphs.

Theorem 2.8. *If X and Y are two subsets of the vertex set V of an Eulerian directed graph D, then we have*

$$\left| |E(X, Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(D)} \right| \leq \sigma_0 \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(D)}.$$

Proof. Let $\mathbf{1}_X$ be the indicator variable of X, i.e., $\mathbf{1}_X(u) = 1$ if $u \in X$ and $\mathbf{1}_X(u) = 0$ otherwise. We define $\mathbf{1}_Y$ similarly. Assume $\mathbf{1}_X T^{1/2} = a_0 \phi_0 + a_1 \phi_1$ and $\mathbf{1}_Y T^{1/2} = b_0 \phi_0 + b_1 \phi_2$, where ϕ_1, ϕ_2 are unit vectors in $\phi_0^\perp$. Since ϕ_0 is a unit vector, we have

$$a_0 = \left\langle \mathbf{1}_X T^{1/2}, \phi_0 \right\rangle = \frac{\mathrm{vol}(X)}{\sqrt{\mathrm{vol}(D)}} \tag{2.5}$$

and

$$a_0^2 + a_1^2 = \left\langle \mathbf{1}_X T^{1/2}, \mathbf{1}_X T^{1/2} \right\rangle = \mathrm{vol}(X). \tag{2.6}$$

Thus

$$a_1 = \sqrt{\mathrm{vol}(X)\mathrm{vol}(\bar{X})/\mathrm{vol}(D)}. \tag{2.7}$$

Similarly, we get

$$b_0 = \frac{\mathrm{vol}(Y)}{\sqrt{\mathrm{vol}(D)}} \quad \text{and} \quad b_1 = \sqrt{\mathrm{vol}(Y)\mathrm{vol}(\bar{Y})/\mathrm{vol}(D)}. \tag{2.8}$$

It follows that

$$\left| |E(X, Y)| - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(D)} \right|$$
$$= \left| \mathbf{1}_X T^{1/2}(L_0 - \phi_0'\phi_0)(\mathbf{1}_Y T^{1/2})' \right|$$
$$= |(a_0\phi_0 + a_1\phi_1)(L_0 - \phi_0'\phi_0)(b_0\phi_0 + b_1\phi_2)'| = |a_1 b_1 \phi_1 L_0 \phi_2'|$$
$$\leq |a_1 b_1| \|\phi_1\| \|L_0\phi_2'\| \leq |a_1 b_1|\sigma_0 = \sigma_0 \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(D)}.$$

The proof of the theorem is complete. $\qquad\qquad\square$

If we use $\bar{\lambda}$ instead of σ_0, then we get a weaker theorem on edge expansions. The proof will be omitted, since it is very similar to the proof of Theorem 2.8.

Theorem 2.9. *Let D be an Eulerian directed graph. If X and Y are two subsets of $V(D)$, then we have*

$$\left| \frac{|E(X,Y)| + |E(Y,X)|}{2} - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(D)} \right| \leq \bar{\lambda} \cdot \frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}}{\mathrm{vol}(D)}.$$

For $X, Y \subseteq V(D)$, let $d(X,Y) = \min\{d(u,v) : u \in X \text{ and } v \in Y\}$. We have the following upper bound on $d(X,Y)$.

Theorem 2.10. *Suppose that D is a connected Eulerian directed graph. For $X, Y \subseteq V(D)$ such that $d(X,Y) \geq 2$ and $0 \leq \alpha < 1$, we have*

$$d(X,Y) \leq \left\lfloor \log\sqrt{\frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}} \middle/ \log\sigma_\alpha \right\rfloor + 1.$$

In particular, for $0 \leq \alpha < 1$, the diameter of D satisfies

$$\mathrm{diam}(D) \leq \left\lceil \frac{\log(\mathrm{vol}(D)/\delta)}{\log\sigma_\alpha} \right\rceil,$$

where $\delta = \min\{d_x : x \in V\}$.

Remark 2.11. From Lemma 2.6, we have

$$\sigma_\alpha^2 \leq \alpha^2 + 2\alpha(1-\alpha)\lambda_1 + (1-\alpha)^2\sigma_0^2.$$

We can choose α to minimize σ_α. If $\lambda_1 \leq 1 - \sigma_0^2$, then we choose $\alpha = 0$ and get $\sigma_\alpha = \sigma_0$; if $\lambda_1 > 1 - \sigma_0^2$, then we choose

$$\alpha = \frac{\lambda_1 + \sigma_0^2 - 1}{2\lambda_1 + \sigma_0^2 - 1}$$

and get

$$\sigma_\alpha^2 \leq 1 - \frac{\lambda_1^2}{2\lambda_1 + \sigma_0^2 - 1}.$$

Combining the two cases, we have

$$\min_{0 \leq \alpha < 1}\{\sigma_\alpha\} \leq \begin{cases} \sigma_0 & \text{if } \lambda_1 \leq 1 - \sigma_0^2, \\ \sqrt{1 - \frac{\lambda_1^2}{2\lambda_1 + \sigma_0^2 - 1}} & \text{otherwise.} \end{cases} \tag{2.9}$$

It is easy to check that

$$\min_{0\leq\alpha<1}\{\sigma_\alpha\} \leq \sqrt{1-\frac{\lambda_1}{2}}.$$

Here the inequality is strict if $\sigma_0 < 1$. We have

$$d(X,Y) \leq \left\lfloor \log\frac{\text{vol}(\bar{X})\text{vol}(\bar{Y})}{\text{vol}(X)\text{vol}(Y)} \middle/ \log\frac{2}{2-\lambda_1} \right\rfloor + 1.$$

Theorem 2.10 is stronger than Theorem 2.5 in general.

Proof. Similar to what we did in the proof of Theorem 2.8, let $\mathbf{1}_X$ and $\mathbf{1}_Y$ be the indicator functions of X and Y, respectively. We have

$$\mathbf{1}_X T^{1/2} = a_0\phi_0 + a_1\phi_1 \quad \text{and} \quad \mathbf{1}_Y T^{1/2} = b_0\phi_0 + b_1\phi_2,$$

where ϕ_1, ϕ_2 are unit vectors in $\phi_0^\perp$, and a_0, b_0, a_1, b_1 are given by (2.5) through (2.8).

Let

$$k = \left\lfloor \log\sqrt{\frac{\text{vol}(\bar{X})\text{vol}(\bar{Y})}{\text{vol}(X)\text{vol}(Y)}} \middle/ \log\sigma_\alpha \right\rfloor + 1.$$

We have

$$\left(\mathbf{1}_X T^{1/2}\right) L_\alpha^k \left(\mathbf{1}_Y T^{1/2}\right)' \geq a_0 b_0 + \sigma_\alpha^k a_1 b_1 > 0.$$

Thus there is a directed path starting from some vertex in X and ending at some vertex in Y, that is, $d(X,Y) \leq k$.

For the diameter result, we choose $X = \{x\}$ and $Y = \{y\}$. Note that $\text{vol}(X) = d_x \geq \delta$, $\text{vol}(Y) = d_y \geq \delta$, $\text{vol}(\bar{X}) < \text{vol}(G)$, and $\text{vol}(\bar{y}) < \text{vol}(G)$. The result follows. $\qquad\square$

3.   Definition of the sth Laplacian

Let H be an r-uniform hypergraph (or an r-graph for short) with vertex set $V(H)$ (or V for short) and edge set $E(H)$. We assume that $|V(H)| = n$ and $E(H) \subseteq \binom{V}{r}$. For a vertex subset S such that $|S| < r$, the *neighborhood* $\Gamma(S)$ is $\{T \mid S \cap T = \varnothing \text{ and } S \cup T \text{ is an edge in } H\}$. Let the *degree* d_S of S in H be the number of edges containing S, i.e., $d_S = |\Gamma(S)|$. For $1 \leq s \leq r-1$, an s-walk of length k is a sequence of vertices

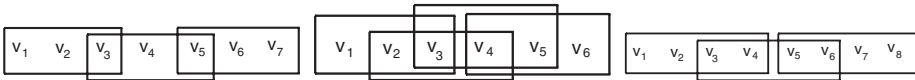$$v_1, v_2, \ldots, v_j, \ldots, v_{(r-s)(k-1)+r}$$

Figure 1. Three examples of an s-walk in a hypergraph: a 1-walk in a 3-graph (left), a 2-walk in a 3-graph (center), and a 2-walk in a 4-graph right).

together with a sequence of edges $F_1, F_2, \ldots, F_k$ such that

$$F_i = \big\{v_{(r-s)(i-1)+1}, v_{(r-s)(i-1)+2}, \ldots, v_{(r-s)(i-1)+r}\big\}$$

for $1 \leq i \leq k$. Some examples of s-walks are shown in Figure 1.

For each i in $\{0, 1, \ldots, k\}$, the *ith stop* x_i of an s-walk is the ordered s-tuple $\big(v_{(r-s)i+1}, v_{(r-s)i+2}, \ldots, v_{(r-s)i+s}\big)$. The initial stop is x_0, and the terminal stop is x_k. An s-walk is an *s-path* if stops (as ordered s-tuples) are distinct. If $x_0 = x_k$, then an s-walk is *closed*.

An *s-cycle* is a special closed s-walk such that $v_1, v_2, \ldots, v_{(r-s)k}$ are distinct and $v_{(r-s)k+j} = v_j$ for $1 \leq j \leq s$ (see Figure 2). An s-cycle is a *loose* cycle if $s \leq r/2$ (particularly $s = 1$); an s-cycle is a *tight* cycle if $s > r/2$ (particularly $s = r - 1$). An s-cycle is *Hamiltonian* if it covers each vertex in H exactly once. In the literature, Hamiltonian tight cycles were first studied in [Katona and Kierstead 99]. Hamiltonian s-cycles for a full range of s were studied in [Rödl et al. 06, Rödl et al. 08, Keevash et al. 12, Kühn et al. 12, Kühn and Osthus 06, Hàn and Schacht 99]. Hamiltonian s-cycles in random r-uniform hypergraphs were studied in [Dudek and Frieze 11, Dudek and Frieze 12, Frieze 10].

For $1 \leq s \leq r - 1$ and $x, y \in V^{\underline{s}}$, the *$s$-distance* $d^{(s)}(x, y)$ is the minimum integer k such that there exists an s-path of length k starting from x and ending at y. A hypergraph H is *s-connected* if $d^{(s)}(x, y)$ is finite for every pair (x, y). If H is s-connected, then the *s-diameter* of H is the maximum value of $d^{(s)}(x, y)$ for $x, y \in V^{\underline{s}}$.
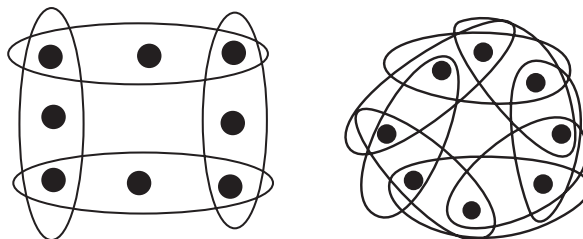

Figure 2. Examples of a loose cycle and a tight cycle in a 3-graph: a 1-cycle in a 3-graph (left) and a 2-cycle in a 3-graph (right).

A random s-walk with initial stop x_0 is an s-walk generated as follows. Let x_0 be the sequence of visited vertices at the initial step. At each step, let S be the set of the last s vertices in the sequence of visited vertices. A random $(r-s)$-set T is chosen from $\Gamma(S)$ uniformly; the vertices in T are added into the sequence one by one in an arbitrary order.

For $0 \le \alpha \le 1$, an α-lazy random s-walk is a modified random s-walk such that with probability α, one can stay at the current stop; with probability $1 - \alpha$, append $r - s$ vertices to the sequence as selected in a random s-walk.

For $x \in V^{\underline{s}}$, let $[x]$ be the s-set consisting of the coordinates of x.

3.1. Case $1 \le s \le r/2$

For $1 \le s \le r/2$, we define a weighted undirected graph $G^{(s)}$ over the vertex set $V^{\underline{s}}$ as follows. Let the weight $w(x, y)$ be $|\{F \in E(H) : [x] \sqcup [y] \subseteq F\}|$. Here $[x] \sqcup [y]$ is the disjoint union of $[x]$ and $[y]$. In particular, if $[x] \cap [y] \ne \varnothing$, then $w(x, y) = 0$.

For $x \in V^{\underline{s}}$, the degree of x in $G^{(s)}$, denoted by $d_x^{(s)}$, is given by

$$d_x^{(s)} = \sum_y w(x, y) = d_{[x]} \binom{r - s}{s} s!. \tag{3.1}$$

Here $d_{[x]}$ means the degree of the set $[x]$ in the hypergraph H. When we restrict an s-walk on H to its stops, we get a walk on $G^{(s)}$. This restriction preserves the length of the walk. Therefore, the s-distance $d^{(s)}(x, y)$ in H is simply the graph distance between x and y in $G^{(s)}$; the s-diameter of H is simply the diameter of the graph $G^{(s)}$.

A random s-walk on H is essentially a random walk on $G^{(s)}$. It can be constructed from a random walk on $G^{(s)}$ by inserting $r - 2s$ additional random vertices T_i between two consecutive stops x_i and x_{i+1} at time i, where T_i is chosen uniformly from $\Gamma([x_i] \cup [x_{i+1}])$ and the vertices T_i are inserted between x_i and x_{i+1} in an arbitrary order.

Therefore, we define the *sth Laplacian* $\mathcal{L}^{(s)}$ of H to be the Laplacian of the weighted undirected graph $G^{(s)}$.

The eigenvalues of $\mathcal{L}^{(s)}$ are listed as $\lambda_0^{(s)}, \lambda_1^{(s)}, \ldots, \lambda_{\binom{n}{s}s!-1}^{(s)}$ in nondecreasing order. Let $\lambda_{\max}^{(s)} = \lambda_{\binom{n}{s}s!-1}^{(s)}$ and $\bar{\lambda}^{(s)} = \max\{|1 - \lambda_1^{(s)}|, |1 - \lambda_{\max}^{(s)}|\}$. For some hypergraphs, the numerical values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ are shown in Table 1.

H	$\lambda_1^{(4)}$	$\lambda_1^{(3)}$	$\lambda_1^{(2)}$	$\lambda_1^{(1)}$	$\lambda_{\max}^{(1)}$	$\lambda_{\max}^{(2)}$	$\lambda_{\max}^{(3)}$	$\lambda_{\max}^{(4)}$
K_6^3			3/4	6/5	6/5	3/2		
K_7^3			7/10	7/6	7/6	3/2		
K_6^4		1/3	5/6	6/5	6/5	3/2	1.76759	
K_7^4		3/8	9/10	7/6	7/6	7/5	7/4	
K_6^5	0.1464	1/2	5/6	6/5	6/5	3/2	3/2	1.809
K_7^5	0.1977	5/8	9/10	7/6	7/6	7/5	3/2	1.809

Table 1.   The values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ of some complete hypergraphs K_n^r.

3.2.   The case $r/2 < s \le r-1$

For $r/2 < s \le r-1$, we define a directed graph $D^{(s)}$ over the vertex set $\mathrm{V}^{\underline{s}}$ as follows. For $x, y \in \mathrm{V}^{\underline{s}}$ such that $x = (x_1, \ldots, x_s)$ and $y = (y_1, \ldots, y_s)$, let (x, y) be a directed edge if $x_{r-s+j} = y_j$ for $1 \le j \le 2s - r$ and $[x] \cup [y]$ is an edge of H.

For $x \in \mathrm{V}^{\underline{s}}$, the out-degree d_x^+ in $D^{(s)}$ and the in-degree d_x^- in $D^{(s)}$ satisfy

$$d_x^+ = d_{[x]}(r - s)! = d_x^-.$$

Thus $D^{(s)}$ is an Eulerian directed graph. We write $d_x^{(s)}$ for both d_x^+ and d_x^-. Now $D^{(s)}$ is strongly connected if and only if it is weakly connected.

Note that an s-walk on H can be naturally viewed as a walk on $D^{(s)}$ and vice versa. Thus the s-distance $d^{(s)}(x, y)$ in H is exactly the directed distance from x to y in $G^{(s)}$; the s-diameter of H is the diameter of $D^{(s)}$. A random s-walk on H is in one-to-one correspondence with a random walk on $D^{(s)}$.

For $r/2 < s \le r-1$, we define the *sth Laplacian* $\mathcal{L}^{(s)}$ as the Laplacian of the Eulerian directed graph $D^{(s)}$ (see Section 2).

The eigenvalues of $\mathcal{L}^{(s)}$ are listed as $\lambda_0^{(s)}, \lambda_1^{(s)}, \ldots, \lambda_{\binom{n}{s}s!-1}^{(s)}$ in nondecreasing order. Let $\lambda_{\max}^{(s)} = \lambda_{\binom{n}{s}s!-1}^{(s)}$ and $\bar{\lambda}^{(s)} = \max\{|1 - \lambda_1^{(s)}|, |1 - \lambda_{\max}^{(s)}|\}$. For some hypergraphs, the numerical values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ are shown in Table 1.

3.3.   Examples

Let K_n^r be the complete r-uniform hypergraph on n vertices. Here we compute the values of $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ for some K_n^r (see Table 1).

From Table 1, we observe that $\lambda_1^{(s)} = \lambda_{\max}^{(s)}$ for some complete hypergraphs. In fact, this is true for every complete hypergraph K_n^r. We have the following property.

Property 3.1. *For an r-uniform hypergraph H and an integer s such that $1 \leq s \leq r/2$, we have $\lambda_1^{(s)}(H) = \lambda_{\max}^{(s)}(H)$ if and only if $s = 1$ and H is a 2-design.*

Proof. In one direction, suppose $s = 1$ and H is a 2-design. For each pair of vertices, the number of edges containing the pair is a constant. Thus $G^{(s)}$ is a complete weighted graph. The Laplacian of any complete weighted graph is the same as the Laplacian of the complete graph K_n. Thus, $\lambda_1^{(s)}(H) = \lambda_{\max}^{(s)}(H)$.

In the other direction, suppose $\lambda_1^{(s)}(H) = \lambda_{\max}^{(s)}(H) = \lambda$. We have

$$\mathcal{L}^{(s)} = \lambda I - \lambda \phi_0^* \phi_0, \tag{3.2}$$

where ϕ_0 is the unit eigenvector corresponding to the trivial eigenvalue 0. Taking the trace of both sides, we get

$$\binom{n}{s} s! = \lambda \binom{n}{s} s! - \lambda.$$

Solving for λ, we get

$$\lambda = 1 + \frac{1}{\binom{n}{s} s! - 1}.$$

Write $\mathcal{L}^{(s)} = I - T^{-1/2} A T^{-1/2}$, where A is the weight matrix of $G^{(s)}$ and T is the diagonal matrix of $d_x^{(s)}$ in $G^{(s)}$. From (3.2), we have

$$T^{-1/2} A T^{-1/2} = -\frac{1}{\binom{n}{s} s! - 1} I + \frac{\binom{n}{s} s!}{\binom{n}{s} s! - 1} \phi_0^* \phi_0. \tag{3.3}$$

If $s \geq 2$, then some off-diagonal entries of A are zero by the definition of $G^{(s)}$. However, all off-diagonal entries of the right-hand-side matrix are nonzero, a contradiction. We must have $s = 1$. In this case, we have

$$\phi_0 = \frac{1}{\mathrm{vol}\left(G^{(1)}\right)} \left(\sqrt{d_1}, \ldots, \sqrt{d_n}\right)^*.$$

By comparing the diagonal entries of (3.3), we get $d_1 = d_2 = \cdots = d_n$. Thus H is a 2-design. $\qquad \square$

4. Properties of Laplacians

In this section, we prove some properties of the Laplacians for hypergraphs.

Lemma 4.1. *For $1 \leq s \leq r/2$, we have the following properties:*

1. *The sth Laplacian has $\binom{n}{s}s!$ eigenvalues, and all of them are in $[0,2]$.*

2. *The number of 0 eigenvalues is the number of connected components in $G^{(s)}$.*

3. *The Laplacian $\mathcal{L}^{(s)}$ has an eigenvalue 2 if and only if $r = 2s$ and $G^{(s)}$ has a bipartite component.*

Proof. We have the following facts (see [Chung 97, Chapter 1]) for a weighted graph G: all Laplacian eigenvalues of G are in $[0,2]$; the number of 0 eigenvalues is the number of connected components in G; G has an eigenvalue 2 if and only if G has a bipartite component.

Items 1 and 2 follow from the facts of the Laplacian of $G^{(s)}$. We thus need to prove only item 3. If $\mathcal{L}^{(s)}$ has an eigenvalue 2, then $G^{(s)}$ has a bipartite component T. We want to show that $r = 2s$. Suppose $r \geq 2s + 1$. Let $\{v_0, v_1, \ldots, v_{r-1}\}$ be an edge in T. For $0 \leq i \leq 2s + 1$ and $0 \leq j \leq s - 1$, let $g(i,j) = is + j \mod (2s + 1)$ and $x_i = (v_{g(i,0)}, \ldots, v_{g(i,s-1)})$. Observe that $[x_i] \cap [x_{i+1}] = \varnothing$ for all $0 \leq i \leq 2s$ and $x_{2s+1} = x_0$. Thus the sequence $x_0, x_1, \ldots, x_{2s}$ forms an odd cycle in $G^{(s)}$, a contradiction. $\qquad\square$

The following lemma compares $\lambda_1^{(s)}$ and $\lambda_{\max}^{(s)}$ for different s.

Lemma 4.2. *Suppose that H is an r-uniform hypergraph. We have*

$$\lambda_1^{(1)} \geq \lambda_1^{(2)} \geq \cdots \geq \lambda_1^{(\lfloor r/2 \rfloor)}; \tag{4.1}$$

$$\lambda_{\max}^{(1)} \leq \lambda_{\max}^{(2)} \leq \cdots \leq \lambda_{\max}^{(\lfloor r/2 \rfloor)}. \tag{4.2}$$

Remark 4.3. We do not know whether similar inequalities hold for $s > r/2$.

Proof. Let T_s be the diagonal matrix of degrees in $G^{(s)}$, and let $R^{(s)}(f)$ be the Rayleigh quotient of $\mathcal{L}^{(s)}$. It suffices to show that $\lambda_1^{(s)} \leq \lambda_1^{(s-1)}$ for $2 \leq s \leq r/2$. Recall that $\lambda_1^{(s)}$ can be defined via the Rayleigh quotient; see (2.1). Pick a function $f : V^{(s-1)} \to R$ such that $\langle f, T_{s-1}\mathbf{1}\rangle = 0$ and $\lambda_1^{(s-1)} = R^{(s-1)}(f)$. We define $g : V^{\underline{s}} \to R$ as follows:

$$g(x) = f(x'),$$

where x' is an $(s-1)$-tuple consisting of the first $(s-1)$ coordinates of x with the same order in x. Applying (3.1), we get

$$\langle g, T_s\mathbf{1}\rangle = \sum_{x \in V^{\underline{s}}} d_x^{(s)} g(x) = \sum_{x \in V^{\underline{s}}} g(x) d_{[x]} \binom{r-s}{s} s!.$$

We have

$$\sum_x g(x) d_{[x]} = \sum_x \sum_{F:[x] \subseteq F} g(x)$$

$$= \sum_{x'} \sum_{F:[x'] \subseteq F} (r - s + 1) f(x') = \sum_{x'} d_{[x']} (r - s + 1) f(x')$$

$$= \frac{r - s + 1}{\binom{r-s+1}{s-1}(s-1)!} \sum_{x'} f(x') d_{x'}^{(s-1)} = 0.$$

Here the second-to-last equality follows from (3.1), and the last one follows from the choice of f. Therefore,

$$\sum_x g(x) d_x^{(s)} = (r - s + 2)(r - s + 1) \sum_{x'} f(x') d_{x'}^{(s-1)}.$$

Thus $\langle g, T_s \mathbf{1} \rangle = 0$. Similarly, we have

$$\sum_x g(x)^2 d_x^{(s)} = (r - s + 2)(r - s + 1) \sum_{x'} f(x')^2 d_{x'}^{(s-1)}.$$

Putting these together, we obtain

$$\sum_x g(x)^2 d_x^{(s)} = (r - s + 2)(r - s + 1) \sum_{x'} f(x')^2 d_{x'}^{(s-1)}.$$

By a similar counting method, we have

$$\sum_{x \sim y} (g(x) - g(y))^2 w(x, y) = \sum_{x \sim y} \sum_{F:[x] \sqcup [y] \subseteq F} (g(x) - g(y))^2$$

$$= \sum_{x' \sim y'} \sum_{F:[x'] \sqcup [y'] \subseteq F} (r - s + 1)(r - s + 2)(f(x') - f(y'))^2$$

$$= (r - s + 1)(r - s + 2) \sum_{x' \sim y'} (f(x') - f(y'))^2 w(x', y').$$

Thus $R^{(s)}(g) = R^{(s-1)}(f) = \lambda_1^{(s-1)}$ by the choice of f. Since $\lambda_1^{(s)}$ is the infimum over all g, we get $\lambda_1^{(s)} \leq \lambda_1^{(s-1)}$.

The inequality (4.2) can be proved similarly. Since $\lambda_{\max}^{(s)}$ is the supremum of the Rayleigh quotient, the direction of the inequalities is reversed. $\qquad \square$

Lemma 4.4. *For $r/2 < s \leq r - 1$, we have the following facts:*

1. *The sth Laplacian has $\binom{n}{s} s!$ eigenvalues, and all of them are in $[0, 2]$.*

2. *The number of 0 eigenvalues is the number of strongly connected components in $D^{(s)}$.*

3. If 2 *is an eigenvalue of* $\mathcal{L}^{(s)}$*, then one of the s-connected components of* H *is bipartite.*

The proof is trivial and will be omitted.

5. Applications

In this section, we show some applications of Laplacians $\mathcal{L}^{(s)}$ of hypergraphs.

5.1.   Random *s*-Walks on Hypergraphs

For $0 \leq \alpha < 1$ and $1 \leq s \leq r/2$, after restricting an α-lazy random s-walk on a hypergraph H to its stops (see Section 3), we get an α-lazy random walk on the corresponding weighted graph $G^{(s)}$. Let $\pi(x) = d_x/\mathrm{vol}(V^{\underline{s}})$ for all $x \in V^{\underline{s}}$, where d_x is the degree of x in $G^{(s)}$ and $\mathrm{vol}(V^{\underline{s}})$ is the volume of $G^{(s)}$. Applying Theorem 2.1, we have the following theorem.

Theorem 5.1.   *For* $1 \leq s \leq r/2$*, suppose that* H *is an s-connected r-uniform hypergraph and* $\lambda_1^{(s)}$ *is the first nontrivial eigenvalue of the sth Laplacian of* H*, while* $\lambda_{\max}^{(s)}$ *is the last. For* $0 \leq \alpha < 1$*, the joint distribution* f_k *at the kth stop of an* α*-lazy random walk at time* k *converges to the stationary distribution* π *in probability. In particular, we have*

$$\left\| (f_k - \pi) T^{-1/2} \right\| \leq \left(\bar{\lambda}_\alpha^{(s)} \right)^k \left\| (f_0 - \pi) T^{-1/2} \right\|,$$

where

$$\bar{\lambda}_\alpha^{(s)} = \max \left\{ |1 - (1-\alpha)\lambda_1^{(s)}|, \ |(1-\alpha)\lambda_{\max}^{(s)} - 1| \right\},$$

and f_0 *is the probability distribution at the initial stop.*

For $0 < \alpha < 1$ and $r/2 < s \leq r - 1$, when restricting an α-lazy random s-walk on a hypergraph H to its stops (see Section 2), we get an α-lazy random walk on the corresponding directed graph $D^{(s)}$. Let $\pi(x) = d_x/\mathrm{vol}(V^{\underline{s}})$ for all $x \in V^{\underline{s}}$, where d_x is the degree of x in $D^{(s)}$ and $\mathrm{vol}(V^{\underline{s}})$ is the volume of $D^{(s)}$. Applying Theorem 2.7, we have the following theorem.

Theorem 5.2.   *For* $r/2 < s \leq r - 1$*, suppose that* H *is an s-connected r-uniform hypergraph and* $\lambda_1^{(s)}$ *is the first nontrivial eigenvalue of the sth Laplacian of* H*. For* $0 < \alpha < 1$*, the joint distribution* f_k *at the kth stop of an* α*-lazy random walk*

at time k converges to the stationary distribution π in probability. In particular, we have

$$\left\| (f_k - \pi) T^{-1/2} \right\| \leq (\sigma_\alpha^{(s)})^k \left\| (f_0 - \pi) T^{-1/2} \right\|,$$

where $\sigma_\alpha^{(s)} \leq \sqrt{1 - 2\alpha(1-\alpha)\lambda_1^{(s)}}$, and f_0 is the probability distribution at the initial stop.

Remark 5.3. The reason that we require $0 < \alpha < 1$ in the case $r/2 < s \leq r - 1$ is that $\sigma_0(D^{(s)}) = 1$ for $r/2 < s \leq r - 1$.

5.2. The s-Distances and s-Diameters in Hypergraphs

Let H be an r-uniform hypergraph. For $1 \leq s \leq r - 1$ and $x, y \in \mathrm{V}^{\underline{s}}$, the s-distance $d^{(s)}(x, y)$ is the minimum integer k such that there is an s-path of length k starting at x and ending at y. For $X, Y \subseteq \mathrm{V}^{\underline{s}}$, let

$$d^{(s)}(X, Y) = \min \left\{ d^{(s)}(x, y) \mid x \in X, y \in Y \right\}.$$

If H is s-connected, then the s-diameter $\mathrm{diam}^{(s)}(H)$ satisfies

$$\mathrm{diam}^{(s)}(H) = \max_{x, y \in \mathrm{V}^{\underline{s}}} \left\{ d^{(s)}(x, y) \right\}.$$

For $1 \leq s \leq r/2$, the s-distances in H and the s-diameter of H are simply the respective graph distances in $G^{(s)}$ and diameter of $G^{(s)}$. Applying Theorem 2.2 and Corollary 2.3, we have the following theorems.

Theorem 5.4. *Suppose H is an r-uniform hypergraph. For integer s such that $1 \leq s \leq r/2$, let $\lambda_1^{(s)}$ be the first nontrivial eigenvalue of the sth Laplacian of H, and $\lambda_{\max}^{(s)}$ the last. Suppose $\lambda_{\max}^{(s)} > \lambda_1^{(s)} > 0$. For $X, Y \subseteq \mathrm{V}^{\underline{s}}$, if $d^{(s)}(X, Y) \geq 2$, then we have*

$$d^{(s)}(X, Y) \leq \left\lceil \log \sqrt{\frac{\mathrm{vol}(\bar{X})\mathrm{vol}(\bar{Y})}{\mathrm{vol}(X)\mathrm{vol}(Y)}} \middle/ \log \frac{\lambda_{\max}^{(s)} + \lambda_1^{(s)}}{\lambda_{\max}^{(s)} - \lambda_1^{(s)}} \right\rceil.$$

Here $\mathrm{vol}(*)$ are volumes in $G^{(s)}$.

Remark 5.5. We know that $\lambda_1^{(s)} > 0$ if and only if H is s-connected. The condition $\lambda_{\max}^{(s)} > \lambda_1^{(s)}$ holds unless $s = 1$ and every pair of vertices is covered by edges evenly (i.e., H is a 2-design).

Theorem 5.6. *Suppose H is an r-uniform hypergraph. For integer s such that $1 \leq s \leq r/2$, let $\lambda_1^{(s)}$ be the first nontrivial eigenvalue of the sth Laplacian of H, and*

let $\lambda_{\max}^{(s)}$ be the last. If $\lambda_{\max}^{(s)} > \lambda_1^{(s)} > 0$, then the s-diameter of an r-uniform hypergraph H satisfies

$$\operatorname{diam}^{(s)}(H) \leq \left\lceil \log \frac{\operatorname{vol}(V^{\underline{s}})}{\delta^{(s)}} \bigg/ \log \frac{\lambda_{\max}^{(s)} + \lambda_1^{(s)}}{\lambda_{\max}^{(s)} - \lambda_1^{(s)}} \right\rceil.$$

Here

$$\operatorname{vol}(V^{\underline{s}}) = \sum_{x \in V^{\underline{s}}} d_x = |E(H)| \frac{r!}{(r - 2s)!},$$

and $\delta^{(s)}$ is the minimum degree in $G^{(s)}$.

When $r/2 < s \leq r - 1$, the s-distances in H and the s-diameter of H are respectively the directed distance in $D^{(s)}$ and the diameter of $D^{(s)}$. Applying Theorem 2.10 and Remark 2.11, we have the following theorems.

Theorem 5.7. *Let H be an r-uniform hypergraph. For $r/2 < s \leq r - 1$ and $X, Y \subseteq V^{\underline{s}}$, if H is s-connected, then we have*

$$d^{(s)}(X, Y) \leq \left\lfloor \log \frac{\operatorname{vol}(\bar{X})\operatorname{vol}(\bar{Y})}{\operatorname{vol}(X)\operatorname{vol}(Y)} \bigg/ \log \frac{2}{2 - \lambda_1^{(s)}} \right\rfloor + 1.$$

Here $\lambda_1^{(s)}$ is the first nontrivial eigenvalue of the Laplacian of $D^{(s)}$, and $\operatorname{vol}(*)$ are volumes in $D^{(s)}$.

Theorem 5.8. *For $r/2 < s \leq r - 1$, suppose that an r-uniform hypergraph H is s-connected. Let $\lambda_1^{(s)}$ be the smallest nonzero eigenvalue of the Laplacian of $D^{(s)}$. The s-diameter of H satisfies*

$$\operatorname{diam}^{(s)}(H) \leq \left\lceil 2 \log \frac{\operatorname{vol}(V^{\underline{s}})}{\delta^{(s)}} \bigg/ \log \frac{2}{2 - \lambda_1^{(s)}} \right\rceil.$$

Here $\operatorname{vol}(V^{\underline{s}}) = \sum_{x \in V^{\underline{s}}} d_x = |E(H)| r!$ and $\delta^{(s)}$ is the minimum degree in $D^{(s)}$.

5.3. Edge Expansions in Hypergraphs

In this subsection, we prove some results on edge expansions in hypergraphs. Note that there has been some attempt to generalize the edge discrepancy theorem from graphs to hypergraphs [Butler unpubl.].

Let H be an r-uniform hypergraph. For $S \subseteq \binom{V}{s}$, we recall that the volume of S satisfies

$$\mathrm{vol}(S) = \sum_{x \in S} d_x.$$

Here d_x is the degree of the set x in H. In particular, we have

$$\mathrm{vol}\left(\binom{V}{s}\right) = |E(H)|\binom{r}{s}.$$

The *density* $e(S)$ of S is $\mathrm{vol}(S)/\mathrm{vol}(\binom{V}{s})$. Let $\bar{S}$ be the complement of S in $\binom{V}{s}$. We have

$$e(\bar{S}) = 1 - e(S).$$

For $1 \leq t \leq s \leq r - t$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let

$$E(S,T) = \{F \in E(H) : \exists x \in S, \exists y \in T, x \cap y = \varnothing, \text{ and } x \cup y \subseteq F\}.$$

Note that $|E(S,T)|$ counts the number of edges contained in $x \sqcup y$ for some $x \in S$ and $y \in T$. In particular, we have

$$\left| E\left(\binom{V}{s}, \binom{V}{t}\right) \right| = |E(H)| \frac{r!}{s!t!(r-s-t)!}.$$

Theorem 5.9. *For $1 \leq t \leq s \leq r/2$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let*

$$e(S,T) = \frac{|E(S,T)|}{|E(\binom{V}{s}, \binom{V}{t})|}.$$

We have

$$|e(S,T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}. \qquad (5.1)$$

Proof. Let $G^{(s)}$ be the weighed undirected graph defined in Section 3. Define S' and T' (sets of ordered s-tuples) as follows:

$$S' = \{x \in V^{\underline{s}} \mid [x] \in S\}, \quad T' = \{(y,z) \in V^{\underline{s}} \mid [y] \in T\}.$$

Let $\bar{S}'$ and $\bar{T}'$ be the respective complements of S' and T' in $V^{\underline{s}}$. We make a convention that $\mathrm{vol}_{G^{(s)}}(*)$ denotes volumes in $G^{(s)}$, while $\mathrm{vol}(*)$ denotes volumes

in H. We have

$$\text{vol}_{G^{(s)}}(G^{(s)}) = \text{vol}\left(\binom{V}{s}\right)\frac{s!(r-s)!}{(r-2s)!}; \tag{5.2}$$

$$\text{vol}_{G^{(s)}}(S') = \text{vol}(S)\frac{s!(r-s)!}{(r-2s)!}; \tag{5.3}$$

$$\text{vol}_{G^{(s)}}(T') = \text{vol}(T)\frac{t!(r-t)!}{(r-2s)!}; \tag{5.4}$$

$$\text{vol}_{G^{(s)}}(\bar{S}') = \text{vol}(\bar{S})\frac{s!(r-s)!}{(r-2s)!}; \tag{5.5}$$

$$\text{vol}_{G^{(s)}}(\bar{T}') = \text{vol}(\bar{T})\frac{t!(r-t)!}{(r-2s)!}. \tag{5.6}$$

Let $E_{G^{(s)}}(S', T')$ be the number of edges between S' and T' in $G^{(s)}$. We get

$$|E_{G^{(s)}}(S', T')| = \frac{(r-s-t)!s!t!}{(r-2s)!}|E(S,T)|.$$

Applying Theorem 2.4 to the sets S' and T' in $G^{(s)}$, we obtain

$$\left||E_{G^{(s)}}(S',T')| - \frac{\text{vol}_{G^{(s)}}(S')\text{vol}_{G^{(s)}}(T')}{\text{vol}_{G^{(s)}}(G^{(s)})}\right|$$
$$\leq \bar{\lambda}_1^{(s)} \frac{\sqrt{\text{vol}_{G^{(s)}}(S')\text{vol}_{G^{(s)}}(T')\text{vol}_{G^{(s)}}(\bar{S}')\text{vol}_{G^{(s)}}(\bar{T}')}}{\text{vol}_{G^{(s)}}(G^{(s)})}.$$

Combining (5.2) through (5.6) and the inequality above, we obtain inequality (5.1). $\qquad\square$

Now we consider the case that $s > r/2$. Due to the fact that $\sigma_0^{(s)} = 1$, we have to use the weaker expansion theorem, Theorem 2.9. Note that

$$\left|E\left(\binom{V}{s}, \binom{V}{t}\right)\right| = |E(H)|\frac{r!}{(r-s-t)!s!t!}.$$

We get the following theorem.

Theorem 5.10. *For $1 \leq t < r/2 < s < s+t \leq r$, $S \subseteq \binom{V}{s}$, and $T \subseteq \binom{V}{t}$, let*

$$e(S,T) = \frac{|E(S,T)|}{|E(\binom{V}{s}, \binom{V}{t})|}.$$

If $|x \cap y| \neq \min\{t, 2s-r\}$ for every $x \in S$ and $y \in T$, then we have

$$\left|\frac{1}{2}e(S,T) - e(S)e(T)\right| \leq \bar{\lambda}^{(s)}\sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}. \tag{5.7}$$

Proof. Recall that $D^{(s)}$ is the directed graph defined in Section 3. Let

$$S' = \{x \in V^{\underline{s}} \mid [x] \in S\},$$
$$T' = \{(y, z) \in V^{\underline{s}} \mid [z] \in T\}.$$

We also denote by $\bar{S}'$ and $\bar{T}'$ the respective complements of S' and T' in $V^{\underline{s}}$. We use the convention that $\text{vol}_{D^{(s)}}(*)$ denotes volumes in $D^{(s)}$, while $\text{vol}(*)$ denotes volumes in the hypergraph H. We have

$$\text{vol}_{D^{(s)}}(D^{(s)}) = \text{vol}\left(\binom{V}{s}\right) s!(r-s)!, \tag{5.8}$$

$$\text{vol}_{D^{(s)}}(S') = \text{vol}(S)s!(r-s)!, \tag{5.9}$$

$$\text{vol}_{D^{(s)}}(T') = \text{vol}(T)t!(r-t)!, \tag{5.10}$$

$$\text{vol}_{D^{(s)}}(\bar{S}') = \text{vol}(\bar{S})s!(r-s)!, \tag{5.11}$$

$$\text{vol}_{D^{(s)}}(\bar{T}') = \text{vol}(\bar{T})s!(r-s)!. \tag{5.12}$$

Let $E_{D^{(s)}}(S', T')$ be the number of directed edges from S' to T' in $D^{(s)}$, and let $E_{D^{(s)}}(T', S')$ be the number of such edges from T' to S'. We get

$$|E_{D^{(s)}}(S', T')| = (r - s - t)!s!t!|E(S, T)|.$$

From the condition $|x \cap y| \neq \min\{t, 2s - r\}$ for each $x \in S$ and each $y \in T$, we observe that

$$E_{D^{(s)}}(T', S') = 0.$$

Applying Theorem 2.9 to the sets S' and T' in $D^{(s)}$, we obtain

$$\left| \frac{|E_{D^{(s)}}(S', T')| + |E_{D^{(s)}}(T', S')|}{2} - \frac{\text{vol}_{D^{(s)}}(S')\text{vol}_{D^{(s)}}(T')}{\text{vol}_{D^{(s)}}(D^{(s)})} \right|$$
$$\leq \bar{\lambda}_1^{(s)} \frac{\sqrt{\text{vol}_{D^{(s)}}(S')\text{vol}_{D^{(s)}}(T')\text{vol}_{D^{(s)}}(\bar{S}')\text{vol}_{D^{(s)}}(\bar{T}')}}{\text{vol}_{D^{(s)}}(D^{(s)})}.$$

Combining (5.8) through (5.12) and the inequality above, we get inequality (5.7). □

Nevertheless, we have the following strong edge expansion theorem for $r/2 < s \leq r - 1$. For $S, T \subseteq \binom{V}{s}$, let $E'(S, T)$ be the set of edges of the form $x \cup y$ for some $x \in S$ and $y \in T$. Namely,

$$E'(S, T) = \{F \in E(H) \mid \exists x \in S, \exists y \in T, F = x \cup y\}.$$

Observe that

$$\left| E'\left(\binom{V}{s}, \binom{V}{s}\right) \right| = |E(H)| \frac{r!}{(r-s)!(2s-r)!(r-s)!}.$$

Theorem 5.11. *For $r/2 < s \leq r - 1$ and $S, T \subseteq \binom{V}{s}$, let*

$$e'(S,T) = \frac{|E'(S,T)|}{|E'(\binom{V}{s}, \binom{V}{s})|}.$$

We have

$$|e'(S,T) - e(S)e(T)| \leq \bar{\lambda}^{(s)} \sqrt{e(S)e(T)e(\bar{S})e(\bar{T})}. \tag{5.13}$$

Proof. Let

$$S' = \{x \in V^{\underline{s}} \mid [x] \in S\},$$
$$T' = \{y \in V^{\underline{s}} \mid [y[\in T\}.$$

Let $\bar{S}'$ and $\bar{T}'$ be the respective complements of S' and T' in $V^{\underline{s}}$. We use the convention that $\mathrm{vol}_{D^{(s)}}(*)$ denotes volumes in $D^{(s)}$, while $\mathrm{vol}(*)$ denotes volumes in the hypergraph H. We have

$$\mathrm{vol}_{D^{(s)}}(D^{(s)}) = \mathrm{vol}\left(\binom{V}{s}\right) s!(r-s)!; \tag{5.14}$$

$$\mathrm{vol}_{D^{(s)}}(S') = \mathrm{vol}(S)s!(r-s)!; \tag{5.15}$$
$$\mathrm{vol}_{D^{(s)}}(T') = \mathrm{vol}(T)s!(r-s)!; \tag{5.16}$$
$$\mathrm{vol}_{D^{(s)}}(\bar{S}') = \mathrm{vol}(\bar{S})s!(r-s)!; \tag{5.17}$$
$$\mathrm{vol}_{D^{(s)}}(\bar{T}') = \mathrm{vol}(\bar{T})s!(r-s)!. \tag{5.18}$$

Let $E_{D^{(s)}}(S', T')$ and $E_{D^{(s)}}(T', S')$ be the respective numbers of directed edges from S' to T' and from T' to S' in $D^{(s)}$. We get

$$|E_{D^{(s)}}(S', T')| = |E_{D^{(s)}}(T', S')| = (r-s)!(2s-r)!(r-s)!|E'(S,T)|.$$

Applying Theorem 2.9 to the sets S' and T' on $D^{(s)}$, we obtain

$$\left| \frac{|E_{D^{(s)}}(S', T')| + |E_{D^{(s)}}(T', S')|}{2} - \frac{\mathrm{vol}_{D^{(s)}}(S')\mathrm{vol}_{D^{(s)}}(T')}{\mathrm{vol}_{D^{(s)}}(D^{(s)})} \right|$$
$$\leq \bar{\lambda}_1^{(s)} \frac{\sqrt{\mathrm{vol}_{D^{(s)}}(S')\mathrm{vol}_{D^{(s)}}(T')\mathrm{vol}_{D^{(s)}}(\bar{S}')\mathrm{vol}_{D^{(s)}}(\bar{T}')}}{\mathrm{vol}_{D^{(s)}}(D^{(s)})}.$$

Combining (5.14) through (5.18) and the inequality above, we get inequality (5.13). $\qquad\square$

6.   Concluding Remarks

In this paper, we introduced a set of Laplacians for r-uniform hypergraphs. For $1 \leq s \leq r - 1$, the s-Laplacian $\mathcal{L}^{(s)}$ is derived from the random s-walks on

hypergraphs. For $1 \leq s \leq r/2$, the sth Laplacian $\mathcal{L}^{(s)}$ is defined to be the Laplacian of the corresponding weighted graph $G^{(s)}$. The first Laplacian $\mathcal{L}^{(1)}$ is exactly the Laplacian introduced in [Rodríguez 09].

For $r/2 \leq s \leq r-1$, the sth Laplacian $\mathcal{L}^{(s)}$ is defined to be the Laplacian of the corresponding Eulerian directed graph $D^{(s)}$. From Lemma 2.6, Theorem 2.7, and Theorem 2.8, it seems that $\sigma_0(D^{(s)})$ might be a good parameter. However, it is not hard to show that $\sigma_0(D^{(s)}) = 1$ always holds, which makes Theorem 2.8 useless for hypergraphs. We can use the weaker Theorem 2.9 for hypergraphs. Our work is based on (with some improvements) the recent work [Chung 05, Chung 06] on directed graphs.

Let us recall Chung's definition of Laplacians [Chung 93] for regular hypergraphs. An r-uniform hypergraph H is d-regular if $d_x = d$ for every $x \in V^{\underline{r-1}}$. Let G be a graph on the vertex set $V^{\underline{r-1}}$. For $x, y \in V^{\underline{r-1}}$, let xy be an edge if $x = x_1 x_2, \ldots, x_{r-1}$ and $y = y_1 x_2, \ldots, x_{r-1}$ such that $\{x_1, y_1, x_2, \ldots, x_{r-1}\}$ is an edge of H. Let A be the adjacency matrix of G, T the diagonal matrix of degrees in G, and K the adjacency matrix of the complete graph on the edge set $V^{\underline{r-1}}$. In [Chung 93], the Laplacian $\mathcal{L}$ is defined such that

$$\mathcal{L} = T - A + \frac{d}{n}(K + (r-1)I).$$

This definition comes from the homology theory of hypergraphs [Chung 93]. Firstly, $\mathcal{L}$ is not normalized in Chung's definition, i.e., the eigenvalues are not in the interval $[0, 2]$. Secondly, the add-on term

$$\frac{d}{n}(K + (r-1)I)$$

is not related to the structures of H. If we ignore the add-on term and normalize the matrix, then we essentially get the Laplacian of the graph G. Note that if G is disconnected, then $\lambda_1(G) = 0$, and this situation is not interesting. Thus Chung added an additional term. The graph G is actually very close to our Eulerian directed graph $D^{(r-1)}$. Let B be the adjacency matrix of $D^{(r-1)}$. In fact, we have $B = QA$, where Q is a rotation that maps $x = x_1, x_2, \ldots, x_{r-1}$ to $x' = x_2, \ldots, x_{r-1}, x_1$. Since $d_x = d_{x'}$, it follows that Q and T commute, and we have

$$(T^{-1/2}BT^{-1/2})'(T^{-1/2}BT^{-1/2}) = T^{-1/2}B'T^{-1}BT^{-1/2}$$
$$= T^{-1/2}A'Q'T^{-1}QAT^{-1/2} = T^{-1/2}A'T^{-1}Q'QAT^{-1/2}$$
$$= T^{-1/2}A'T^{-1}AT^{-1/2}.$$

Here we use the fact that $Q'Q = I$. This identity means that the singular values of $I - \mathcal{L}^{(r-1)}$ are precisely equal to 1 minus the Laplacian eigenvalues of the graph G.

Our definitions of Laplacians $\mathcal{L}^{(s)}$ seem to be related to the quasirandomness [Chung and Graham 90, Kohayakawa et al. 02] of hypergraphs. We are very interested in this direction. Many concepts such as the s-walk, the s-path, the s-distance, and the s-diameter are of independent interest.

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