*Internet Mathematics*, 11:134–142, 2015

ISSN: 1542-7951 print/1944-9488 online
DOI: 10.1080/15427951.2014.884513

# TRAFFIC CONGESTION IN EXPANDERS AND $(p, \delta)$–HYPERBOLIC SPACES

**Shi Li[1] and Gabriel H. Tucci[2]**
[1] *Toyota Technological Institute at Chicago, Chicago, Illinois, USA*
[2] *Barclays Capital Inc., New York, New York, USA*

**Abstract** *In this article we define the notion of $(p, \delta)$–Gromov hyperbolic space where we relax the Gromov slimness condition to allow that not all, but a positive fraction of all triangles, are $\delta$–slim. Furthermore, we study their traffic congestion under geodesic routing. We also construct a constant degree family of expanders with congestion $\Theta(n^2)$ in contrast to random regular graphs that have congestion $O(n \log^3(n))$.*

## 1. INTRODUCTION

The purpose of this work is to continue the study of traffic congestion under geodesic routing. By geodesic routing we mean that the path chosen to route the traffic between the nodes is the shortest path. We assume there is a predefined, consistent way to break ties so that the shortest path between any two nodes is uniquely defined. Our set up throughout the article is as follows. Let $\{G_n\}_{n=1}^{\infty}$ be a family of connected graphs where $G_n$ has $n$ nodes. For each pair of nodes in $G_n$, consider a unit flow of traffic that travels through the shortest path between nodes as previously discussed. Hence, the total traffic flow in $G_n$ is equal to $n(n-1)/2$. Given a node $v \in G_n$ we define $\mathcal{L}_n(v)$ as the total flow passing through the node $v$. Let $M_n$ be the maximum vertex flow across the network

$$M_n := \max \Big\{ \mathcal{L}_n(v) \; : \; v \in G_n \Big\}.$$

It is easy to see that for any graph $n - 1 \leq M_n \leq n(n-1)/2$.

It was observed experimentally in [16], and proved formally in [1, 2], that if the family is Gromov hyperbolic, then the maximum vertex congestion scales as $\Theta(n^2)$. More precisely, let $\{G_n\}_{n=1}^{\infty}$ be an increasing sequence of finite simple graphs such that $|G_n| = n$, and consider traffic flowing in these graphs that is uniform and geodesically routed between all pairs of nodes. If this sequence is uniformly Gromov $\delta$–hyperbolic, for some nonnegative $\delta$, then there is a sequence of nodes $\{x_n\}_{n=1}^{\infty}$, with $x_n \in G_n$, such that the total traffic passing through $x_n$ is greater than $cn^2$ for some positive constant $c$ independent of $n$. These highly congested nodes are called the *core* of the graph. For more details and related results see [3–15].

Address correspondence to Gabriel H. Tucci, Barclays Capital Inc., 745 Seventh Avenue, New York, NY 10019, USA. E-mail: gabriel.tucci@barclays.com

In this work we extend the previously discussed work and study what happens with the traffic congestion if we relax the slimness condition so that not all, but a fraction of all the triangles, are $\delta$–slim. More precisely, we say that a metric $(X, d)$ is $(p, \delta)$–hyperbolic if for at least a $p$ fraction of the 3–tuples $(u, v, w) \in X \times X \times X$, the geodesic triangle $\triangle_{uvw}$ is $\delta$–slim. The case $p$ equals 1 corresponds to the classic Gromov $\delta$–hyperbolic space. We show that the congestion in these graphs scales as $\Omega(p^2n^2/D_n^2)$, where $D_n$ is the diameter of $G_n$.

Another important family of graphs are expanders. In graph theory, an expander graph is a sparse graph that has strong connectivity properties. Expander constructions have spawned research in pure and applied mathematics, with several applications to complexity theory, design of robust computer networks, and the theory of error-correcting codes. It is well known that random regular graphs are a large family of expanders. It was proved in [17] that random $d$–regular graphs have maximum vertex congestion scaling as $O(n \log^3_{d-1}(n))$. Therefore, it is a natural question to ask if expanders always have low congestion under geodesic routing. In Section 4, we show that this is not the case. More precisely, we construct a family of expanders $\{G_n\}_{n=1}^{\infty}$ with maximum vertex congestion $\Theta(n^2)$.

## 2. PRELIMINARIES

### 2.1. Hyperbolic Metric Spaces

In this section we review the notion of Gromov $\delta$–hyperbolic space. There are many equivalent definitions of Gromov hyperbolicity but the one we take as our definition is the property that triangles are *slim*.

**Definition 2.1.** Let $\delta$. be a nonnegative number, a geodesic triangle in a metric space $X$ is said to be $\delta$–slim if each of its sides is contained in the $\delta$–neighborhood of the union of the other two sides. A geodesic space $X$ is said to be $\delta$–hyperbolic if every triangle in $X$ is $\delta$–slim. A space is said to be hyperbolic if it is $\delta$-hyperbolic for some nonnegative $\delta$.

It is easy to see that any tree is 0-hyperbolic. Other examples of hyperbolic spaces include finite graphs, the fundamental group of a surface of genus greater than or equal to 2, the classical hyperbolic space, and any regular tessellation of the hyperbolic space (i.e., infinite planar graphs with uniform degree $q$ and $p$–gons as faces with $(p - 2)(q - 2) > 4$).

### 2.2. Expanders

We say that a family of graphs $\{G_n\}_{n=1}^{\infty}$ is a $c$–expander family if the edge expansion (also an isoperimetric number or Cheeger constant) $h(G_n) \geq c$ where

$$h(G_n) = \min \left\{ \frac{|\partial S|}{|S|} \ : \ S \subset G_n \text{ with } 1 \leq |S| \leq |G_n|/2 \right\}$$

and $\partial S$ is the edge boundary of $S$, i.e., the set of edges with exactly one endpoint in $S$.

## 3. CONGESTION ON $(p, \delta)$–HYPERBOLIC GRAPHS

In this section we generalize the definition of Gromov hyperbolic spaces to include spaces where not all, but a fixed proportion of the triangles in the metric space, are *slim*. Furthermore, we study the traffic characteristics in these graphs.

**Definition 3.1.** $(p, \delta)$**-hyperbolic** We say that a metric space $(X, d)$ is $(p, \delta)$–hyperbolic if, for at least a $p$ fraction of the 3–tuples $(u, v, w) \in X \times X \times X$, the geodesic triangle $\triangle_{uvw}$ is $\delta$–slim.

A classical $\delta$–hyperbolic space corresponds to $p$ equals one.

**Theorem 3.2.** *Let $(X, d)$ be a $(p, \delta)$-hyperbolic metric space of size $n$. Let $D$ be its diameter and $M = \max\{|B(u, \delta)| : u \in X\}$ be the maximum number of points in a ball of radius $\delta$. Then there exists a point $a \in X$ such that the traffic passing through $a$ is at least $p^2n^2/(D^2M^3)$.*

Before proving the theorem, we prove the following useful lemma.

**Lemma 3.3.** *Let $G = (U, V, E)$ be a bipartite graph such that $|U| = |V| = n$ and $|E| \geq pn^2$. The edges of $G$ are colored in such a way that every vertex in $U \cup V$ is incident to at most $t$ colors ($u$ is incident to a color $c$ if $u$ is incident to an edge with color $c$). Then, there must be a color that is used by at least $(pn/t)^2$ edges in $E$.*

**Proof.** Define three random variables $A$, $B$, and $C$ as follows. We randomly select an edge $(u, v) \in E$ and let $A = u$, $B = v$ and $C$ be the color of the edge $(u, v)$.

Because $h(A) \leq \log n$, $h(B) \leq \log n$ and $h(A, B) \geq \log(pn^2) = 2\log n - \log(1/p)$, we have $I(A; B) = h(A) + h(B) - h(A, B) \leq \log(1/p)$. ($h$ is the entropy function and $I(A, B)$ is the mutual information between $A$ and $B$.) Moreover, if we know $A = u$, then there can be at most $t$ possible colors for $C$. Thus, we have $h(C|A) \leq \log t$. Similarly, $h(C|B) \leq \log t$. Hence,

$$\begin{aligned} h(C|B) \geq I(C; A|B) &= h(A|B) - h(A|C, B) = h(A) - I(A; B) - h(A|C, B) \\ &\geq h(A) - I(A; B) - h(A|C) = I(A; C) - I(A; B). \end{aligned}$$

Thus, $h(C) = h(C|A) + I(A; C) \leq h(C|A) + h(C|B) + I(A; B) \leq \log(t^2/p)$.

Notice that $|E| \geq pn^2$. The inequality implies that there must be a color that is used by at least $pn^2/(t^2/p) = (pn/t)^2$ edges. □

Now, we proceed to prove the theorem. For any 3 points $u, v, w \in X$, let $c_{uvw}$ be the barycenter of the triangle $\triangle_{uvw}$. By a simple counting argument, there must be a point $w \in X$ such that for at least $p$ fraction of the ordered pairs $(u, v) \in X \times X$, $\triangle_{uvw}$ is $\delta$–slim. We fix such a point $w$ from now on. Define $G = (U = X, V = X, E)$ as follows. For any two vertices $u \in U$, $v \in V$, $(u, v) \in E$ iff $\triangle_{uvw}$ is $\delta$–slim. The color of $(u, v) \in E$ is $c_{uvw}$. Then, $|E| \geq pn^2$. Moreover, if $c$ is the color of $(u, v)$, then $c$ is in the $\delta$-neighborhood of $[uw]$. Thus, any vertex $u$ can be incident to at most $DM$ colors. By Lemma 3.3, there must be a color $c$ that is used by at least $p^2n^2/(DM)^2$ edges in $E$. Notice that for each such edge $(u, v) \in E$, $c$ is in the $\delta$-neighborhood of $[uv]$. Thus, $[uv]$ will contain a vertex in the ball

$B(c, \delta)$. Since $|B(c, \delta)| \leq M$, for some vertex $c' \in B(c, \delta)$ and $p^2n^2/(D^2M^3)$, different ordered pairs $(u, v) \in X \times X$, $[uv]$ contain $c'$.

## 4. TRAFFIC ON EXPANDERS

In this section we construct a constant degree family of expanders with $\Theta(n^2)$ congestion. This result is in contrast to random regular graphs that have congestion $O(n \log^3(n))$ (as shown in [17]).

**Theorem 4.1.** *There exists a family $\{G_n\}_{n=1}^{\infty}$ of constant-degree expanders with congestion $\Theta(n^2)$.*

### 4.1. Construction of the Expander Graph

For an even integer $h$, let $T$ be a tree of depth $h$ defined as follows. Each node in depth 0 to $h/2 - 1$ of $T$ has 3 children, and each node in depth $h/2$ to $h - 1$ has 2 children (root has depth 0 and leaves have depth $h$). Thus, $T$ has exactly $n := 3^{h/2} \times 2^{h/2} = \sqrt{6}^h$ leaves. We use $L(T)$ to denote the set of leaves of $T$. Define $\lambda(d)$ to be the number of leaves in a subtree of $T$ rooted at some vertex of depth $h - d$. Then, we have

$$\lambda(d) = \begin{cases} 2^d & 0 \leq d \leq h/2 \\ 2^{h/2}3^{d-h/2} & h/2 < d \leq h \end{cases}.$$

Construct a graph $G$ as follows. (See Figure 1.) Let $A$ be a degree-3 expander of size $2n$. Let $T_{\mathsf{L}}$ and $T_{\mathsf{R}}$ be two copies of the tree $T$. We create a random matching between the $2n$ leaves of $T_{\mathsf{L}}$ and $T_{\mathsf{R}}$ and the vertices of $A$. We add a *matching edge* between each matched pair of vertices. We also add a vertex $v^*$ that is connected to the roots of $T_{\mathsf{L}}$ and $T_{\mathsf{R}}$. Finally, we scale matching edges and expander edges by a factor of $c$ (i.e., replace those edges with paths of length $c$) for some constant even number $c$ to be determined later.

**Proposition 4.1.** *The graph $G$ is an expander of degree at most* 4 *and size* $\Theta(n)$.

**Proof.** Because replacing edges with paths of length $c$ decreases the expansion by a factor of only $c$, we need to consider only the graph obtained before the scaling operation. Let $G'$ be that graph. For the simplicity of the notation, let $T'$ be the tree rooted at $v^*$ with 2 sub-trees $T_{\mathsf{L}}$ and $T_{\mathsf{R}}$ and $L(T') = L(T_{\mathsf{L}}) \cup L(T_{\mathsf{R}})$. We also use $I(T')$ to denote the set of inner vertices of $T'$.

Assume that $G'$ is not an $0.01\alpha$-expander. Then, let $S \subseteq V(G')$ be a set of size at most $\left|V(G')\right|/2$ such that $E_{G'}(S, V(G') \setminus S) < 0.01\alpha\,|S|$.

We notice the following two facts:

1. If $\left|S \cap I(T')\right| \geq \left|S \cap L(T')\right| + s$ then there must be at least $s$ edges between $S \cap V(T')$ and $V(T') \setminus S$.
2. If $\left|\,\left|S \cap L(T')\right| - |S \cap V(A)|\,\right| \geq s$ then there must be at least $s$ matching edges between $S$ and $V(G') \setminus S$.

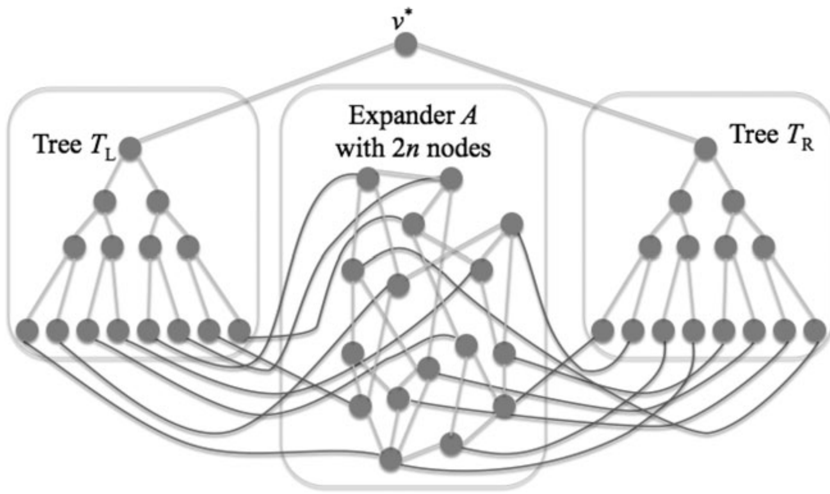


**Figure 1** Construction of the expander $G$. There are two trees $T_\mathsf{L}$ and $T_\mathsf{R}$, each with $n$ leaves, a root node $v^*$, and an expander $A$ with $2n$ nodes. We connect $v^*$ to the two roots of $T_\mathsf{L}$ and $T_\mathsf{R}$. We create a random matching between the $2n$ leaves of $T_\mathsf{L}$ and $T_\mathsf{R}$ and the $2n$ vertices of $A$; there is a path of length $c$ connecting each pair in the matching. We also replace edges of $A$ with paths of length $c$.

By the first fact, we can assume $\left|S \cap I(T')\right| \le 0.6|S|$. Then

$$\left|S \cap L(T')\right| + |S \cap V(A)| \ge 0.4|S|.$$

By the second fact, we have $|S \cap V(A)| \ge 0.15|S|$. If $|S \cap V(A)| \le |V(A)|/2$ then $E_{G'}(S, V(G') \setminus S) \ge 0.15\alpha|S|$. Thus, we have $|S \cap V(A)| \ge |V(A)|/2$. If $|S \cap V(A)| \le 0.9\,|V(A)|$, then

$$E_{G'}(S, V(G') \setminus S) \ge \alpha\,|V(A) \setminus S| \ge \frac{0.1}{0.9}\alpha\,|S \cap V(A)| \ge \alpha|S|/60.$$

Thus, we have $|S \cap V(A)| \ge 0.9\,|V(A)|$, which implies $\left|S \cap L(T')\right| \ge 0.9\,|V(A)| - 0.1|S|$ by the second fact. Then $|S| \ge 1.8\,|V(A)| - 0.1|S|$, implying that

$$|S| \ge 1.6\,|V(A)| \ge \frac{1.6}{3}\left|V(G')\right| > 0.5\left|V(G')\right|,$$

Which, is a contradiction. □

### 4.2. Proof of the High Congestion in $G$

Consider the set $L(T_\mathsf{L}) \times L(T_\mathsf{R})$ of $n^2$ pairs. We show that for a constant fraction of pairs $(u, v)$ in this set, the shortest path connecting $u$ and $v$ will contain $v^*$. It is easy to see that for every $(u, v) \in L(T_\mathsf{L}) \times L(T_\mathsf{R})$, there is a path of length $2h + 2$ connecting $u$ and $v$, which goes through $v^*$.

Focus on the graph $G \setminus v^*$. We are interested in the number of pairs $(u, v) \in L(T_\mathsf{L}) \times L(T_\mathsf{R})$ such that $d_{G\setminus v^*}(u, v) \le 2h + 2$. Fix a vertex $u \in L(T_\mathsf{L})$ from now on. Consider the set $\mathcal{P}$ of simple paths starting at $u$ and ending at $L(T_\mathsf{R})$. We say a path $P \in \mathcal{P}$ has rank $r$ if it enters and leaves the expander $r$ times. Notice that we always have $r \geq 1$, since we must use the expander $A$ from $u$ to $L(T_{\mathsf{R}})$.

For a path $P$ of rank $r$, we define the *pattern* of $P$, denoted by $\mathsf{ptn}(P)$, as a sequence $t = (t_1, t_2, \ldots, t_{2r+1})$ of $2r + 1$ nonnegative integers as follows. For $1 \leq i \leq r$, $ct_{2i}$ is the length of the subpath of $P$ correspondent to the $i$th traversal of $P$ in the expander $A$. (Recall that we replaced each expander edge with a path of length $c$.) The path $P$ will contain $r + 1$ subpaths in the two copies of $T$ (the first and/or the last subpath might have length 0). Let $2t_{2i-1}$ be the length of the $i$th subpath in the tree. Notice that $P$ can enter and leave the trees only through leaves, and thus, the lengths of those subpaths must be even. If some path $P \in \mathcal{P}$ has $\mathsf{ptn}(P) = (t_1, t_2, \ldots, t_{2r+1})$, then the length of $P$ is exactly $\mathsf{len}(t_{[2r+1]}) := 2\sum_{i=0}^{r} t_{2i+1} + c\sum_{i=1}^{r}(t_{2i} + 2)$, where $t_{[2r+1]}$ denotes the sequence $(t_1, t_2, \ldots, t_{2r+1})$. Notice that the $c(t_{2i} + 2)$ term in the definition of $\mathsf{len}$ includes the $2c$ edges replaced by the 2 matching edges through which the path enters and leaves the expander.

We call a sequence $(t_1, t_2, \ldots, t_{2r+1})$ of nonnegative integers *a valid pattern* of rank $r$ if $\mathsf{len}(t_{[2r+1]}) \leq 2h + 2$.

**Lemma 4.2.** *The number of paths $P \in \mathcal{P}$ with $\mathsf{ptn}(P) = t_{[2r+1]}$ is at most*

$$\left(\frac{3}{2}\right)^r \left[\prod_{i=1}^{r} \lambda(t_{2i-1}) 2^{t_{2i}}\right] \lambda(t_{2r+1}).$$

**Proof.** For any leaf $v$ in $T$, we have at most $\lambda(t)$ possible simple paths of length $2t$ in $T$ that start at $v$ and end at $L(T)$. For a degree-3 graph (in particular, a degree-3 expander), we have at most $\frac{3}{2} \times 2^t$ simple paths that start at any fixed vertex. From a leaf in $T_{\mathsf{L}}$ or $T_{\mathsf{R}}$, we have only one way to enter the expander. Similarly, we have only one way to leave the expander from a vertex in the expander. Thus, the total number of simple paths of pattern $t_{[2r+1]}$ is at most

$$\lambda(t_1)\left(\frac{3}{2}2^{t_2}\right)\lambda(t_3)\left(\frac{3}{2}2^{t_4}\right)\cdots\lambda(t_{2r+1}) = \left(\frac{3}{2}\right)^r \left[\prod_{i=1}^{r} \lambda(t_{2i-1}) 2^{t_{2i}}\right] \lambda(t_{2r+1}).$$

$\square$

We fix the rank $r \geq 1$ from now on. For some integer $\ell \in [0, r]$, suppose $t_{[2\ell]}$ is a prefix of some valid pattern of rank $r$. Define $W(t_{[2\ell]})$ to be the maximum $t_{2\ell+1}$ such that $t_{[2\ell+1]}$ is a prefix of some valid pattern of rank $r$. That is,

$$W\left(t_{[2\ell]}\right) = \left\lfloor \frac{2h + 2 - 2\sum_{i=1}^{\ell} t_{2i-1} - c\sum_{i=1}^{\ell}(t_{2i} + 2) - 2c(r - \ell)}{2} \right\rfloor.$$

Similarly, we define

$$W\left(t_{[2\ell-1]}\right) = \left\lfloor \frac{2h + 2 - 2\sum_{i=1}^{\ell} t_{2i-1} - c\sum_{i=1}^{\ell-1}(t_{2i} + 2) - 2c(r - \ell)}{c} \right\rfloor - 2.$$

For some prefix $t_{[2\ell]}$ of some valid pattern of rank $r$, let $\mathcal{P}_{t_{[2\ell]}}$ be the set of paths $P \in \mathcal{P}$ of rank $r$ such that $t_{[2\ell]}$ is a prefix of $\mathsf{ptn}(P)$. We prove that

**Lemma 4.3.** *For any $0 \le \ell \le r$, we have*

$$\left|\mathcal{P}_{t_{[2\ell]}}\right| \le 2C^{r-\ell}\left(\frac{3}{2}\right)^{\ell}\prod_{i=1}^{\ell}\left[\lambda(t_{2i-1})2^{t_{2i}}\right] \times \sqrt{6}^{W\left(t_{[2\ell]}\right)}$$

*for some large enough constant $C$.*

**Proof.** For $\ell = r$, Lemma 4.2 implies

$$\begin{aligned}\left|\mathcal{P}_{t_{[2r]}}\right| &\le \sum_{t_{2r+1}=0}^{W\left(t_{[2r]}\right)}\left(\frac{3}{2}\right)^{r}\prod_{i=1}^{r}\left[\lambda(t_{2i-1})2^{t_{2i}}\right]\lambda(t_{2r+1})\\ &= \left(\frac{3}{2}\right)^{r}\prod_{i=1}^{r}\left[\lambda(t_{2i-1})2^{t_{2i}}\right]\sum_{t_{2r+1}=0}^{W\left(t_{[2r]}\right)}\lambda(t_{2r+1})\\ &\le 2\left(\frac{3}{2}\right)^{r}\prod_{i=1}^{r}\left[\lambda(t_{2i-1})2^{t_{2i}}\right] \times \sqrt{6}^{W\left(t_{[2r]}\right)}.\end{aligned}$$

The last inequality holds because $\sum_{t_{2r+1}=0}^{W\left(t_{[2r]}\right)}\lambda(t_{2r+1}) \le 2\lambda\left(W\left(t_{[2r]}\right)\right) \le 2\sqrt{6}^{W\left(t_{[2r]}\right)}$.

Now, suppose the lemma holds for some $1 \le \ell \le r$ and we shall prove that it holds for $\ell - 1$. By the induction hypothesis, we have

$$\begin{aligned}\left|\mathcal{P}_{t_{[2\ell-2]}}\right| &\le \sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)}\sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)} 2C^{r-\ell}\left(\frac{3}{2}\right)^{\ell}\prod_{i=1}^{\ell}\left[\lambda(t_{2i-1})2^{t_{2i}}\right] \times \sqrt{6}^{W\left(t_{[2\ell]}\right)}\\ &= 2C^{r-\ell}\left(\frac{3}{2}\right)^{\ell}\prod_{i=1}^{\ell-1}\left[\lambda(t_{2i-1})2^{t_{2i}}\right]\sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)}\lambda(t_{2\ell-1})\sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)}2^{t_{2\ell}}\sqrt{6}^{W\left(t_{[2\ell]}\right)}.\end{aligned}$$

It is sufficient to prove that $\sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)}\lambda(t_{2\ell-1})\sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)}2^{t_{2\ell}}\sqrt{6}^{W\left(t_{[2\ell]}\right)} \le \frac{2C}{3}\sqrt{6}^{W\left(t_{[2\ell-2]}\right)}$.

$$\text{LHS} \le \frac{1}{1-2/\sqrt{6}^{c/2}}\sum_{t_{2\ell-1}=0}^{W\left(t_{[2\ell-2]}\right)}\lambda(t_{2\ell-1})\sqrt{6}^{W\left(t_{[2\ell-1]}\bowtie(0)\right)} \tag{4.1}$$

$$\le \frac{1}{1-2/\sqrt{6}^{c/2}}\left(\frac{\sqrt{6}^{W\left(t_{[2\ell-2]}\bowtie(0,0)\right)}}{1-\sqrt{2/3}} + \frac{\lambda\left(W\left(t_{[2\ell-2]}\right)\right)}{1-\sqrt{2/3}}\right) \tag{4.2}$$

$$\le \frac{1}{1-2/\sqrt{6}^{c/2}}\frac{2}{1-\sqrt{2/3}}\sqrt{6}^{W\left(t_{[2\ell-2]}\right)}, \tag{4.3}$$

where $t_{[2\ell-1]} \bowtie (4.1)$ denotes the sequence obtained by concatenating the two sequences $t_{[2\ell-1]}$ and (4.1).

We explain Inequalities (4.1), (4.2), and (4.3) one by one. Focus on the term $Q = 2^{t_{2\ell}}\sqrt{6}^{W\left(t_{[2\ell]}\right)}$. If we increase $t_{2\ell}$ by 1, then $W\left(t_{[2\ell]}\right)$ will decrease by exactly $c/2$, by the definition of $W$. (We assume $c$ is an even number.) Thus, $Q$ will decrease by a factor of $\sqrt{6}^{c/2}/2$. By the rule of the geometric sum,

$$\sum_{t_{2\ell}=0}^{W\left(t_{[2\ell-1]}\right)} Q \le \frac{Q|_{t_{2\ell}=0}}{1-2/\sqrt{6}^{c/2}} = \frac{\sqrt{6}^{W\left(t_{[2\ell-1]}\bowtie(0)\right)}}{1-2/\sqrt{6}^{c/2}},$$

implying Inequality (4.1).

Now focus on the term $Q = \lambda(t_{2\ell-1})\sqrt{6}^{W\left(t_{[2\ell-1]}\bowtie(0)\right)}$. If we increase $t_{2\ell-1}$ by 1, then $W\left(t_{[2\ell-1]} \bowtie (0)\right)$ will decrease by 1. Then, $Q$ will either decrease by a factor of $\sqrt{6}/2 = \sqrt{3/2}$, or increase by a factor of $3/\sqrt{6} = \sqrt{3/2}$, depending on whether $t_{2\ell-1} \le h/2$. We can split the sum $\sum_{t_{2\ell-1}=0}^{W(t_{[2\ell-2]})} Q$ into 2 sums at the point $h/2$ if necessary. Again, using the geometric sum, we have

$$\sum_{t_{2\ell-1}=0}^{W(t_{[2\ell-2]})} Q \le \frac{Q|_{t_{2\ell-1}=0}}{1-\sqrt{2/3}} + \frac{Q|_{t_{2\ell-1}=W(t_{[2\ell-2]})}}{1-\sqrt{2/3}},$$

implying Inequality (4.2).

Inequality (4.3) follows from the fact that $W\left(t_{[2\ell-2]} \bowtie (0,0)\right) = W\left(t_{[2\ell-2]}\right)$ and $\lambda(t) \le \sqrt{6}^{t}$ for any $t \in [0,h]$.

This finishes the proof if we let $C = \dfrac{3}{\left(1-2/\sqrt{6}^{c/2}\right)\left(1-\sqrt{2/3}\right)}$. □

Notice that $W(\emptyset) = \left\lfloor \frac{2h+2-2rc}{2} \right\rfloor = h+1-rc$ for an even integer $c$. Thus,

$$|\mathcal{P}_\emptyset| \le 2C^r\sqrt{6}^{h+1-rc} = 2\sqrt{6}\left(\frac{C}{\sqrt{6}^c}\right)^r \sqrt{6}^h = 2\sqrt{6}\left(\frac{C}{\sqrt{6}^c}\right)^r n.$$

$\mathcal{P}_\emptyset$ is essentially the set of paths in $\mathcal{P}$ with rank $r$. For a large enough $c$, we have $C < \sqrt{6}^c$. Summing up over all $r \ge 1$, we have that the total number of paths in $\mathcal{P}$ is at most

$$\frac{2\sqrt{6}C/\sqrt{6}^c}{1-C/\sqrt{6}^c}n.$$

For a large enough constant $c$ (say, $c = 6$), the number will be at most $n/2$.

Then, consider the graph $G$. We know there is a path of length $2h+2$ from $u$ to $v$ via $v^*$ for any vertex $v \in L(T_{\mathsf{R}})$. Thus, for a fixed vertex $u \in L(T_{\mathsf{L}})$, there are at least $n/2$ vertices $v \in L(T_{\mathsf{R}})$ such that the minimum length between $u$ and $v$ contains $v^*$. Therefore, there are $n^2/2$ pairs $(u,v)$ such that the shortest path between $u$ and $v$ contain $v^*$.

## FUNDING

This work was supported by NIST Grant No. 60NANB10D128.

## REFERENCES


[1] Y. Baryshnikov and G. Tucci. "Asymptotic Traffic Flow in an Hyperbolic Network I: Definition and Properties of the Core." Preprint at http://arxiv.org/abs/1010.3304.

[2] Y. Baryshnikov and G. Tucci. "Asymptotic Traffic Flow in an Hyperbolic Network II: Non-uniform Traffic." Preprint at http://arxiv.org/abs/1010.3305.

[3] S. Buyalo, A. Dranishnikov and V. Schroeder. "Embedding of Hyperbolic Groups into Products of Binary Trees." *Inventiones Mathematicae* 169:1(2007), 153–192.

[4] M. Coornaert. "Measures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov." *Pacific J. Math*. 159:2(1993), 241–270.

[5] M. Coornaert, T. Delzant, and A. Papadopoulos. *Geometrie et theorie des groupes*. Berlin: Springer-Verlag,1993.

[6] W. Floyd. "Group Completions and Limit Sets of Kleinian Groups." *Invent. Math*. 57(1980), 205–218.

[7] E. Ghys and P. de la Harper. *Sur les groupes hyperboliques d'apres Mikhael Gromov*. Boston, MA: Birkhauser Boston, 1990.

[8] M. Gromov. "Hyperbolic Groups." *In Essays in Group Theory*, pp. 75–263. New York, NY: Springer, 1987.

[9] V. Kaimanovich. "The Poisson Formula for Groups with Hyperbolic Properties." Annals of Math. 152(2000), 659–692.

[10] E. Jonckheere, P. Lohsoonthorn, and F. Bonahon. "Scaled Gromov Hyperbolic Graphs." *Journal of Graph Theory* 57(2008), 157–180.

[11] E. Jonckheere, M. Lou, F. Bonahon, and Y. Baryshnikov. "Euclidean Versus Hyperbolic Congestion in Idealized Versus Experimental Networks." http://arxiv.org/abs/0911.2538.

[12] D. Krioukov, F. Papadopoulos, A. Vahdat, and M. Boguna. "Curvature and Temperature of Complex Networks." *Physical Review E* 80:035101(R)(2009).

[13] P. Lohsoonthorn. "Hyperbolic Geometry of Networks." PhD Thesis, Department of Electrical Engineering, University of Southern California, 2003. Available at http://eudoxus.usc.edu/iw/mattfinalthesismain.pdf.

[14] M. Lou. "Traffic Pattern Analysis in Negatively Curved Networks." PhD Thesis, USC, 2008. Available at http://eudoxus.usc.edu/iw/Mingji-PhD-Thesis.pdf.

[15] O. Narayan and I. Saniee. "Scaling of Load in Communication Networks." Physical Review E (statistical physics) 82:036102(2010).

[16] O. Narayan and I. Saniee. "Large-Scale Curvature of Networks." *Physical Review E* (statistical physics), 84:066108(2011). First appeared at http://arxiv.org/0907.1478, 2009.

[17] G. Tucci. "Non Hyperbolicity in Random Regular Graphs and their Traffic Characteristics." To appear in the *Central European Journal of Mathematics*.